Week #10 : Systems of DEs

Goals:

- Introduction to Systems of Differential Equations
- Solving systems with distinct real eigenvalues and eigenvectors

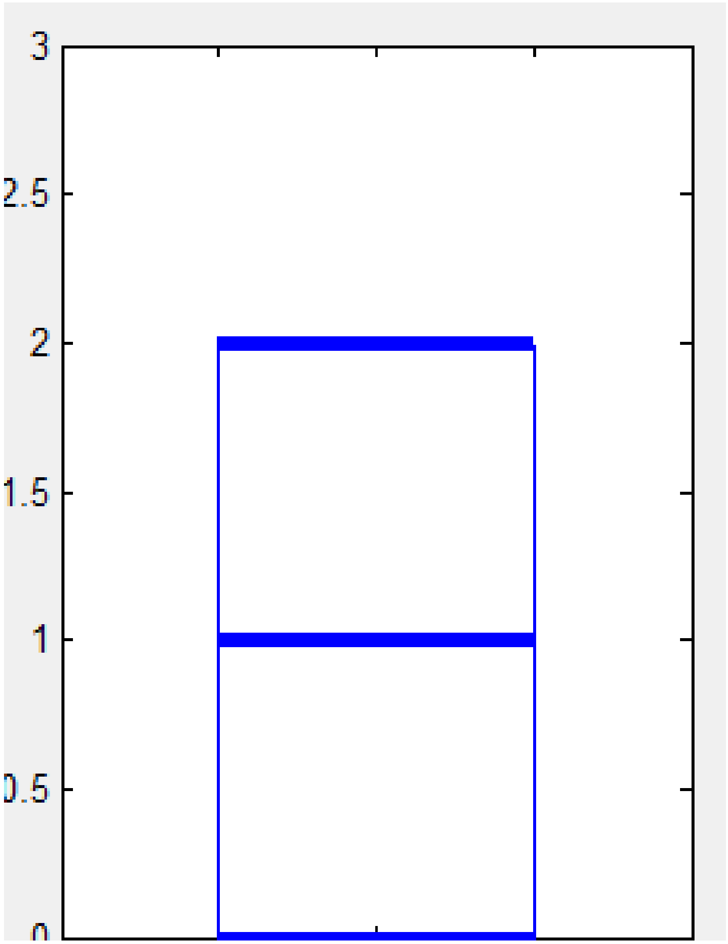
Systems of Differential Equations - Introdution

We have gone about as far as we can with interesting single-variable DEs. In practice, more complex systems involve **multiple, interrelated variables**.

- Complex physical and electronic systems
- Interacting populations like predator/prey and host/parasites

One particularly visceral model is that of a multi-story building in an earthquake.

Demonstration



Model: Spring Models for Buildings

To understand the dynamics in a complex system, we need to go back to basics.

Problem. Draw a double-spring diagram, and a double-column diagram.

If both x_1 and x_2 are increased by an equal amount, what is the force in the second spring/column?

We will use the spring system as our model for the force calculations, simply because it is more familiar, and easier to visualize.

Problem. Draw a free-body diagram for the first mass. Obtain a differential equation for the first mass.

Repeat for the second mass.

Problem. Write out the **system of differential equations** obtained.

Converting Higher-Order DEs to 1st Order Systems

For consistency of analysis, we will transform this **second-order system** into a larger **first-order system**.

Notation: In this section of the course:

- vectors with be written with vector hats, e.g. $\vec{x}$,
- matrices will be written using capital letters, e.g. A or M, and
- scalars will be in lower-case, e.g. c_1, λ.

Problem. Define a new vector of 4 variables, $\vec{w}$, that will allow the conversion of the higher-order system to a first-order system.

Define the derivative of $\vec{w}$ in terms of $\vec{w}$ itself, making use of the DE as necessary.

This is now a **first-order system** of differential equations.
(The variables we are most interested in for this example are $w_1 = x_1$, and $w_3 = x_2$, the *positions* of each mass.)

Problem. Put the equations into matrix format.

Problem. Use a technique from earlier in the course to find a form of the solution.

$$\text{Solving } \vec{w}\,' = A\vec{w}, \text{ assume } \vec{w} = \vec{v}e^{\lambda t}$$

We have reduced our solving of the first-order system of equations to **finding the eigenvalues and eigenvectors of a matrix**.

Problem. If we found eigenvalues $\lambda = 2, 3, 4, 5$, what form would the solution take?

$$\text{Solving } \vec{w}\,' = A\vec{w}, \text{ assume } \vec{w} = \vec{v}e^{\lambda t}$$

Problem. If we found eigenvalues $\lambda = -1 \pm 2i, -2 \pm 3i$, what form would the solution take?

Solving $\vec{w}\,' = A\vec{w}$, assume $\vec{w} = \vec{v}e^{\lambda t}$

Problem. If $\lambda = -1 \pm 2i, -2 \pm 3i$ were the values for our building model in the demonstration software, what would be dangerous frequencies for an external force and why?

Matrices and Linear Systems - Homogeneous Theory

How does linear algebra help solve systems of linear differential equations?

Consider the system of differential equations

$$x_1'(t) = 3x_1(t) - 4x_2(t) \text{ and}$$
$$x_2'(t) = 4x_1(t) - 7x_2(t)$$

Problem. Write this system of equations out in matrix form.

In the matrix/vector form, what kind of rules about the solution can we rely on?

Theorem. *Let $A(t)$ and $\vec{f}(t)$ be continuous on an open interval I. If $t_0 \in I$ and $u \in \mathbb{R}^n$, then there exists a unique solution $\vec{x}(t)$ on I to $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$ where $\vec{x}(t_0) = \vec{x}_0$.*

Theorem. *Let $A(t)$ be a continuous $(n \times n)$-matrix on an open interval I. If $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ are linearly independent solutions to the homogenous system $\vec{x}\,'(t) = A(t)\vec{x}(t)$ on I, then every solution has the form*

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t).$$

Problem. Consider the system of equations

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}(t)$$

Show that the following vector-valued functions are solutions to the system.

(a) $\vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}(t)$$

(b) $\vec{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}$

(c) $\vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}(t)$$

with $\vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$

Problem. Based on the earlier results, write out the **general** solution to the system of equations.

Matrices and Linear Systems - Non-Homogeneous Theory

In analogy to our earlier work, what if we have a system which is **non**-homogeneous?

The standard form for a linear system with a non-homogeneous component is:

$$\vec{x}\,'(t) = A(t)\vec{x}(t) + \vec{f}(t)$$

Theorem. *Let $A(t)$ be a continuous $(n \times n)$-matrix on an open interval I and let $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ be linearly independent solutions to*

$$\vec{x}\,'(t) = A(t)\vec{x}(t) \text{ on } I.$$

If $\vec{x}_{NH}(t)$ satisfies

$$\vec{x}\,'(t) = A(t)\vec{x}(t) + \vec{f}(t) \text{ on } I,$$

then every solution of the nonhomogeneous system has the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) + \vec{x}_{NH}(t).$$

Problem. Verify that $\vec{x}_{NH}(t) = [t-1, \quad -t, \quad t+1]^t$ is a particular solution to

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ -2t-1 \\ 2 \end{bmatrix}$$

Problem. Find the general solution to

$$\vec{x}\,'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ -2t-1 \\ 2 \end{bmatrix}$$

$$\left(\vec{x}_{NH}(t) = \begin{bmatrix} t-1 \\ -t \\ t+1 \end{bmatrix},\ \vec{x}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix},\ \vec{x}_2 = \begin{bmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{bmatrix},\ \vec{x}_3 = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix} \right)$$

DE Systems with Constant Coefficients

Knowing how we can combine individual solutions, how do we find those solutions in the first place?
We saw earlier that **eigenvalues** and **eigenvectors** are tools that could assist us.

Theorem. *Let A be a constant $(n \times n)$-matrix with n linearly independent eigenvectors $\vec{u}_1, \ldots, \vec{u}_n$.*

If r_i is the eigenvalue corresponding to $\vec{u}_i$,

then the general solution to $\vec{x}\,' = A\vec{x}$ is

$$\vec{x}(t) = c_1\ e^{r_1 t}\ \vec{u}_1 + c_2\ e^{r_2 t}\ \vec{u}_2 + \cdots + c_n\ e^{r_n t}\ \vec{u}_n\,.$$

Problem. Solve $\vec{x}\,'(t) = A\vec{x}(t)$ where $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

$$\vec{x}\,'(t) = A\vec{x}(t) \text{ where } A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

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Problem. Verify your solution is correct.

Problem. Solve $\vec{x}\,'(t) = A\vec{x}(t)$ where

$$A := \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \text{ and } \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

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