Week #9 : DEs with Non-Constant Coefficients, Laplace Resonance

Goals:

- Solving DEs with Non-Constant Coefficients
- Resonance with Laplace
- Laplace with Periodic Functions

Solving Equations with Non-Constant Coefficients

Problem. Consider the IVP

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

What techniques from the course could we use to solve this equation?

Proposition (Frequency differentiation)**.** *If* $f(t)$ *is piecewise continuous on* $[0, \infty)$ *and of exponential order* a*, then for* $s > a$ *we have*

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\Big(\mathcal{L}\{f(t)\}(s)\Big) .$$

Problem. Sketch the proof of this relationship.

Problem. Use the general transform

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\Big(\mathcal{L}\{f(t)\}(s)\Big)$$

to compute

$$\mathcal{L}\{tf(t))\}$$

$$\mathcal{L}\{tf(t)\} = -1\frac{d}{ds}\Big(\mathcal{L}\{f(t)\}\Big)$$

Problem. Compute

$$\mathcal{L}\{t\sin(kt)\}$$

IVPs with Non-Constant Coefficients - Example 1

Problem. Use

$$\mathcal{L}\{tf(t)\} = -1\frac{d}{ds}\Big(\mathcal{L}\{f(t)\}\Big)$$

to help solve

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

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Problem. Verify that your solution is correct.

Laplace with 1/t Multipliers - Frequency Integration

A related property can be helpful when $\dfrac{1}{t}$ multipliers are present.

Proposition (Frequency Integration)**.** *Let $f(t)$ be piecewise continuous on $[0,\infty)$, of exponential order a, and $\lim_{t\to 0^+} \dfrac{f(t)}{t}$ is finite. If $F(s) := \mathcal{L}\{f(t)\}(s)$, then we have*

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)\,d\sigma \text{ for } s > a$$

Problem. Compute $\mathcal{L}\left\{\dfrac{\sin(t)}{t}\right\}$.

IVPs with Non-Constant Coefficients - Example 2

Problem. Solve $ty'' + 2y' + ty = 0$, $y(0) = 1$, and $y(\pi) = 0$

Hint. This problem doesn't immediately use our new $1/t$ integration theorem, but wait for it...

$$ty'' + 2y' + ty = 0,\ y(0) = 1, \text{ and } y(\pi) = 0$$

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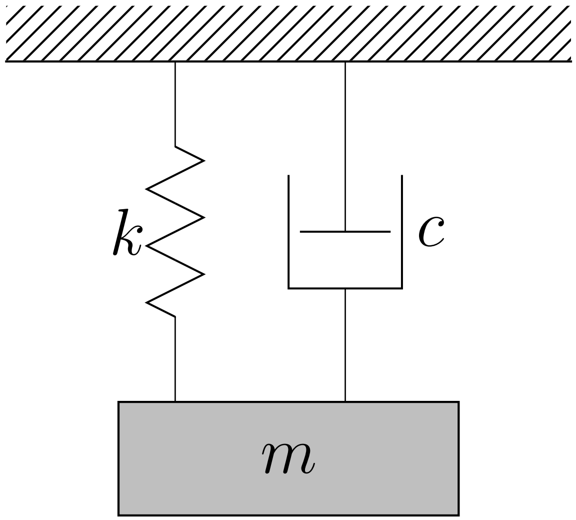
$$ty'' + 2y' + ty = 0,\ y(0) = 1, \text{ and } y(\pi) = 0$$

Problem. Confirm that your solution is correct.

Spring/Mass System Resonance With Laplace



Problem.

Write out the DE for the position of the mass, given $F_{\text{ext}} = F_0 \sin(\omega t)$.

$$my'' + cy' + ky = F_0 \sin(\omega t)$$

Problem. If we set $m = 1$, $c = 0$ and $\omega = \sqrt{k}$ (or $k = \omega^2$), what would this mean for the physical system?

$y'' + \omega^2 y = F_0 \sin(\omega t)$ - no damping, matching frequencies

Problem. Predict the position of the mass over time, given that it starts at equilibrium; use Laplace transforms.

$y'' + \omega^2 y = F_0 \sin(\omega t)$ - no damping, matching frequencies

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Spring/Mass System with Square-Wave Forcing

Problem. For a spring/mass system exhibiting resonance, using $F_{\text{ext}} = F_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$, what element in the external force seems the most relevant to causing resonance?

What other F_{ext} functions might produce the same ever-growing oscillation amplitude?

Problem. Find the natural **period** of the spring/mass system defined by

$$y'' + \frac{\pi^2}{4} y = 0$$

$$y'' + \frac{\pi^2}{4} y = F_{\text{ext}}$$

Problem. Write an F_{ext} function that would push at 1 N for half of a cycle, then nothing for the rest of the cycle, push for a half cycle, then off again, etc.

Problem. Write F_{ext} using step functions.

Problem. Find $\mathcal{L}\{F_{\text{ext}}\}$.

Problem. Predict the motion of the spring/mass system

$$y'' + \frac{\pi^2}{4} y = F_0 \, f\text{sq}(t), \;\; f\text{sq}(t) = \begin{cases} 1 & 0 \le t < 2 \\ 0 & 2 \le t < 4 \\ 1 & 4 \le t < 6 \\ 0 & 6 \le t < 8 \\ \text{etc.} \end{cases}$$

$$y'' + \frac{\pi^2}{4} y = F_0 \; f_{\mathrm{sq}}(t)$$

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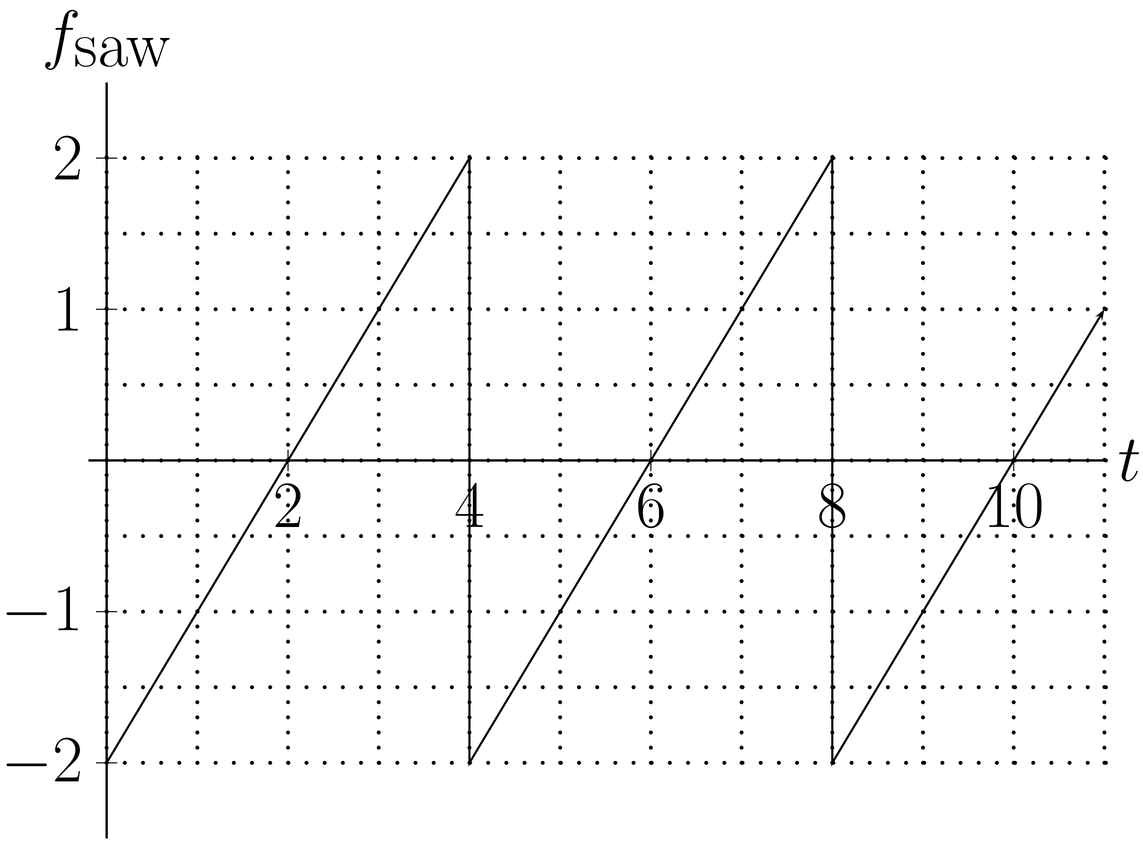
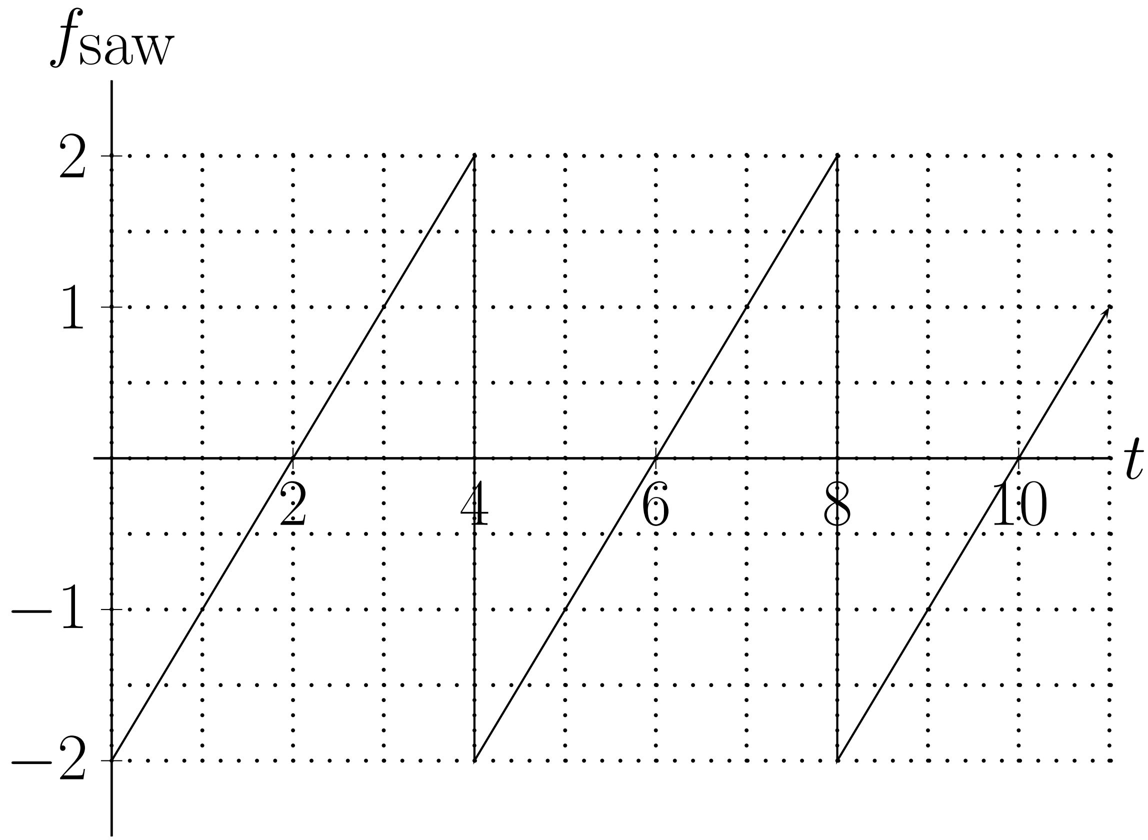
Spring/Mass System with Sawtooth Forcing

We will now study the effect on the spring/mass system using an alternative periodic forcing function, the *sawtooth* function.

Problem. The start of the graph of f_{saw} is shown below. Write out a formula for this periodic function.





Problem. Write f_{saw} using step functions.

Problem. Find $\mathcal{L}\{f_{\text{saw}}\}$.

Problem. Predict the motion of the spring/mass system governed by

$$y'' + \frac{\pi^2}{4} y = F_0 \, f_{\text{saw}}(t)$$

$$y'' + \frac{\pi^2}{4} y = F_0 \, f_{\mathrm{saw}}(t)$$

$$y'' + \frac{\pi^2}{4} y = F_0 \, f_{\mathrm{saw}}(t)$$

Consider an **undamped** spring/mass system.

Problem. If we use a periodic F_{ext} with the same frequency as the natural oscillations, what do you expect to happen?

Problem. If we use a periodic F_{ext} with *almost* the same frequency?

Problem. If we use a periodic F_{ext} with very different frequency?

Now consider an **under**damped spring/mass system.

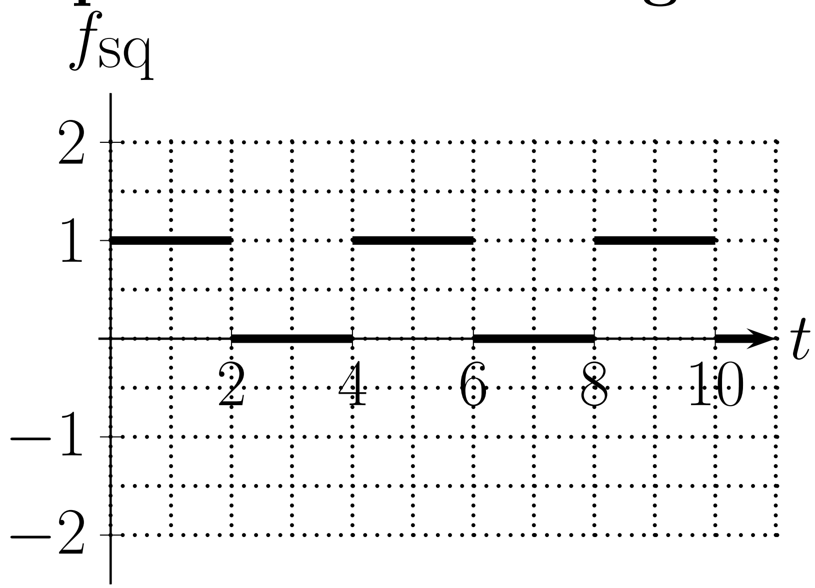
Problem. If we use a periodic F_{ext} with the same frequency as the natural oscillations, what do you expect to happen?

Problem. If we use a periodic F_{ext} with *almost* the same frequency?

Problem. If we use a periodic F_{ext} with very different frequency?

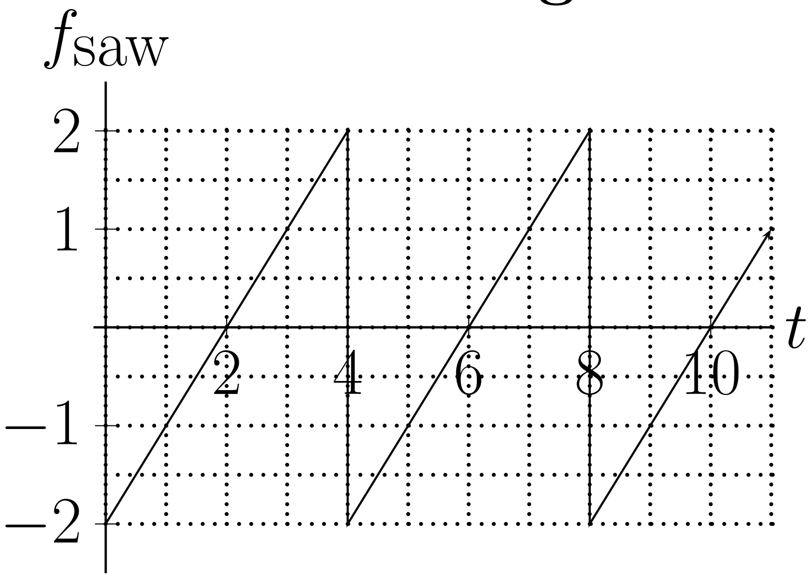
Spring/Mass System Demonstrations

Square-wave Forcing



$$y(t) = F_0 \frac{4}{\pi^2} \begin{cases} 1 - 1\sin\left(\frac{\pi}{2}t\right) & 0 \le t \le 2 \\ -2\sin\left(\frac{\pi}{2}t\right) & 2 \le t \le 4 \\ 1 - 3\sin\left(\frac{\pi}{2}t\right) & 4 \le t \le 6 \\ -4\sin\left(\frac{\pi}{2}t\right) & 6 \le t \le 8 \\ \vdots & \end{cases}$$

Sawtooth Forcing



$$y(t) = F_0 \frac{4}{\pi^2} \begin{cases} t - 2 + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) & 0 \le t \le 4 \\ t - 6 + \frac{2+8}{\pi}\sin\left(\frac{\pi}{2}t\right) & 4 \le t \le 8 \\ t - 10 + \frac{2+16}{\pi}\sin\left(\frac{\pi}{2}t\right) & 8 \le t \le 12 \\ t - 14 + \frac{2+24}{\pi}\sin\left(\frac{\pi}{2}t\right) & 12 \le t \le 16 \\ \vdots \end{cases}$$