Week #7 : Laplace - Step Functions, DE Solutions

Goals:

- Laplace Transform Theory
- Transforms of Piecewise Functions
- Solutions to Differential Equations

Existence of Laplace Transforms

Before continuing our use of Laplace transforms for solving DEs, it is worth digressing through a quick investigation of which functions actually *have* a Laplace transform.

A function f is ***piecewise continuous*** on an interval $t \in [a, b]$ if the interval can be partitioned by a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ such that

- f is continuous on each open subinterval (t_{i-1}, t_i).
- f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is continuous on $[a, b]$ except for a finite number of jump discontinuities. A function is piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

Problem. Draw examples of functions which are continuous and piecewise continuous, or which have different kinds of discontinuities.

One of the requirements for a function having a Laplace transform is that it be piecewise continuous. Classify the graphs above based on this criteria.

Another requirement of the Laplace transform is that the integral $\displaystyle\int_0^\infty e^{-st} f(t)\ dt$ *converges* for at least some values of s. To help determine this, we introduce a generally useful idea for comparing functions, "Big-O notation".

Big-O notation

We write $f(t) = O\big(e^{at}\big)$ as $t \to \infty$ and say f is ***of exponential order*** a (as $t \to \infty$) if there exists a positive real number M and a real number t_0 such that $|f(t)| \le Me^{at}$ for all $t > t_0$.

Lemma. *Assume* $\lim\limits_{t\to\infty} \dfrac{|f(t)|}{e^{at}}$ *exists. Then*

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{at}} < \infty$$

$$\textit{if and only if } f(t) = O\big(e^{at}\big) \textit{ as } t \to \infty. \quad \Box$$

Problem. Show that bounded functions and polynomials are of exponential order a for all $a > 0$.

Problem. Show that e^{t^2} does **not** have exponential order.

Problem. Are all the functions we have seen so far in our DE solutions of exponential order?

The final reveal: what kinds of functions have Laplace transforms?

Proposition. *If f is*

- *piecewise continuous on $[0, \infty)$ and*
- *of exponential order a,*

then the Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for $s > a$.

The proof is based the comparison test for improper integrals.

Laplace Transform of Piecewise Functions

In our earlier DE solution techniques, we could not directly solve non-homogeneous DEs that involved piecewise functions. Laplace transforms will give us a method for handling piecewise functions.

Problem. Use the definition to determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t \leq 5\,, \\ 0 & 5 < t \leq 10\,, \\ e^{4t} & 10 < t\,. \end{cases}$$

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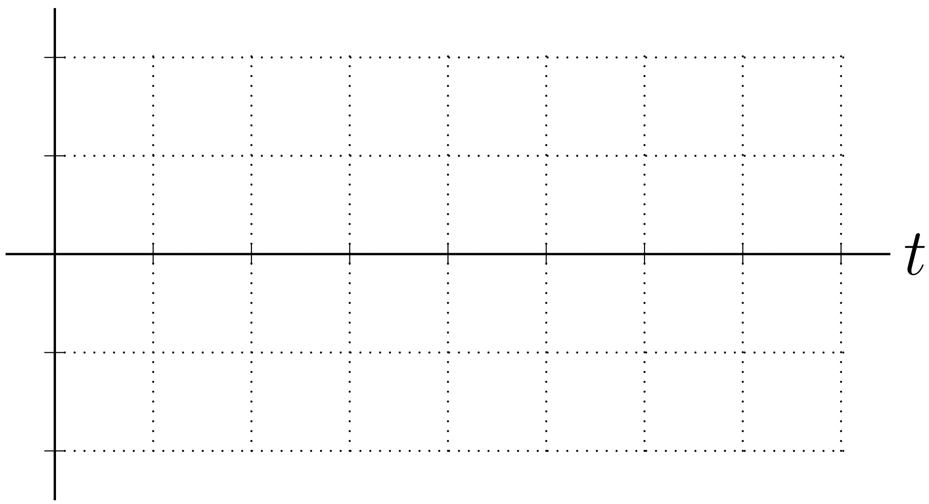
We would like avoid having to use the Laplace definition integral if there is an easier alternative. A new notation tool will help to simplify the transform process.

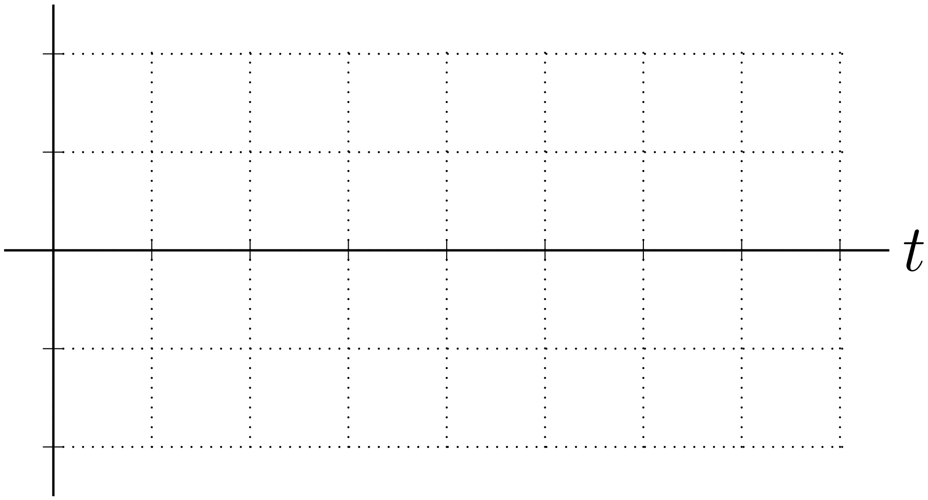
The ***Heaviside step function*** or ***unit step function*** is defined by $u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$

Problem. Sketch the graph of $u(t)$.



$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

Problem. Sketch the graph of $u(t-5)$.



Laplace Transform Using Step Functions

Problem. For $a > 0$, compute the Laplace transform of

$$u(t-a) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$$

Laplace Transform of Step Functions

$$\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$$

An alternate (and more directly useful form) is

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

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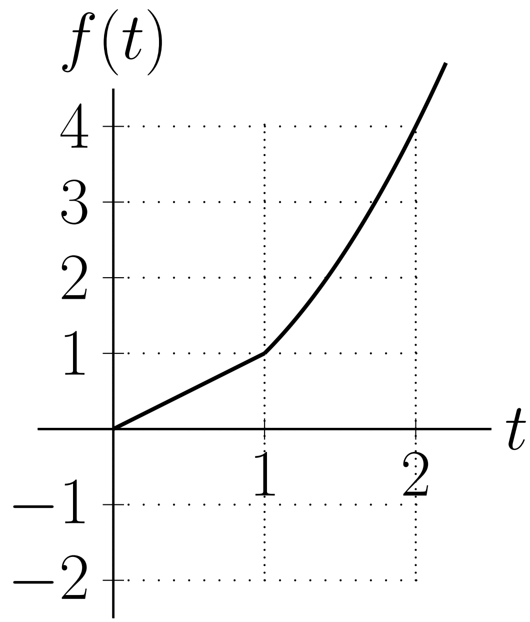
Problem. Find $\mathcal{L}(u_2)$.

Problem. Find $\mathcal{L}(u_\pi)$.

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

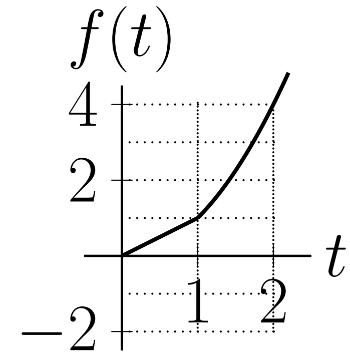
Problem. Find $\mathcal{L}(tu_3)$.

Problem. Here is a more complicated function made up of $f = t$ and $f = t^2$.



Write the function in piecewise form, and again using step functions.

Problem. Find $\mathcal{L}(t(u_0 - u_1) + t^2 u_1)$.



Inverse Laplace Transform of Step Functions

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-4}\right\}$

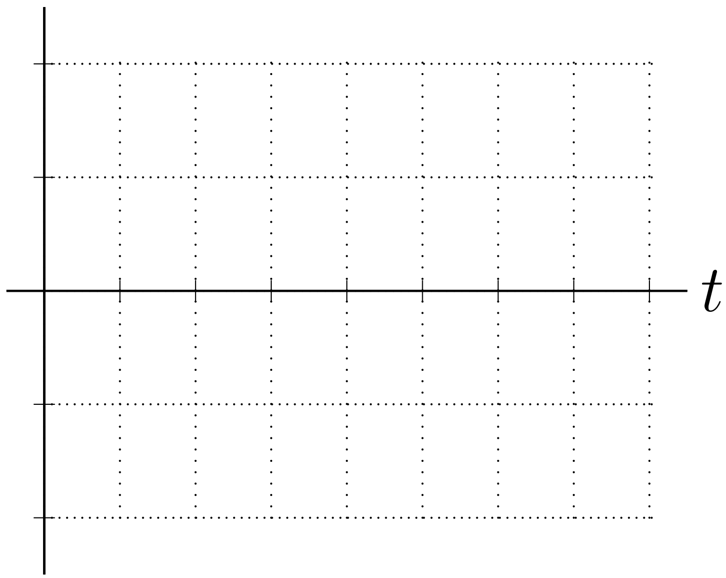
$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Which of the following equals $f(t) = \mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+4}\right\}$?

1. $\dfrac{1}{s}\cos(\pi t)u_\pi$
2. $\dfrac{1}{\pi s}\cos(\pi(t-\pi))u_\pi$
3. $\dfrac{1}{2}\sin(2(t-\pi))u_\pi$
4. $\dfrac{1}{\pi}\sin(2(t-\pi))u_\pi$

Problem. Sketch the graph of

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \frac{1}{2}\sin(2(t-\pi))u_\pi$$



Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-1)(s-2)}\right\}$

Tips for Inverse Laplace With Step/Piecewise Functions

- Separate/group all terms by their e^{-as} factor.

- Complete any partial fractions **leaving the e^{-as} out front** of the term.

  - The e^{-as} only affects final inverse step.

  - Partial fraction decomposition only works for polynomial numerators.

The reason Laplace transforms can be helpful in solving differential equations is because there is a (relatively simple) transform rule for derivatives of functions.

Proposition (Differentiation)**.** *If f is continuous on $[0, \infty)$, $f'(t)$ is piecewise continuous on $[0, \infty)$, and both functions are of exponential order a, then for $s > a$, we have*

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

Problem. Confirm the transform table entry for $\mathcal{L}\{\cos(kt)\}$ with the help of the transform derivative rule and the transform of $\sin(kt)$.

We can generalize this rule to the transform of higher derivatives of a function.

Theorem (General Differentiation)**.** *If* $f(t), f'(t), \ldots, f^{(n-1)}(t)$ *are continuous on* $[0, \infty)$*,* $f^{(n)}(t)$ *is piecewise continuous on* $[0, \infty)$*, and all of these functions are of exponential order* a*, then for* $s > a$*, we have*

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)\,.$$

Can be proven using integration by parts n times.

Most commonly in this course, we will need specifically the transform of the second derivative of a function.

Corollary (Second Differentiation). *If* $f(t)$ *and* $f'(t)$ *are continuous on* $[0, \infty)$, $f''(t)$ *is piecewise continuous on* $[0, \infty)$, *and all of these functions are of exponential order* a, *then for* $s > a$, *we have*

$$\mathcal{L}\{f''(t)\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)\,.$$

Solving Initial Value Problems with Laplace Transforms

Problem. Sketch the general method.

Problem. Find the Laplace transform of the entire DE

$$x' + x = \cos(2t), x(0) = 0$$

Problem. Note the form of the equation now: are there any derivatives left?

Problem. Solve for $X(s)$.

$X(s) =$

Problem. Put $X(s)$ in a form so that you can find its inverse transform.

Problem. Find $x(t)$ by taking the inverse transform.

Problem. Confirm that the function you found is a solution to the differential equation $x' + x = \cos(2t)$.

Problem. Solve $y'' + y = \sin(2t)$, $y(0) = 2$, and $y'(0) = 1$.

$$y'' + y = \sin(2t),\ y(0) = 2, \text{ and } y'(0) = 1.$$

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$$y'' + y = \sin(2t),\ y(0) = 2, \text{ and } y'(0) = 1.$$

Problem. Confirm your solution is correct.

Problem. Solve $y'' - 2y' + 5y = -8e^{-t}$, $y(0) = 2$, and $y'(0) = 12$.

$$y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2, \text{ and } y'(0) = 12.$$

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Problem. Confirm your solution is correct.