

## Week #3 : Second-Order Homogeneous DEs

### Goals:

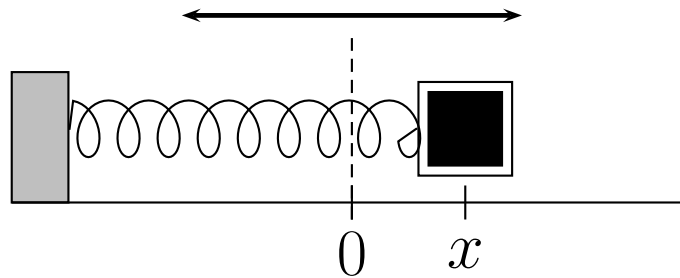
- Second-order Linear Equations
- Linear Independence of Solutions and the Wronskian
- Homogeneous DEs with Constant Coefficients

## Second-Order Linear Equations - Spring System

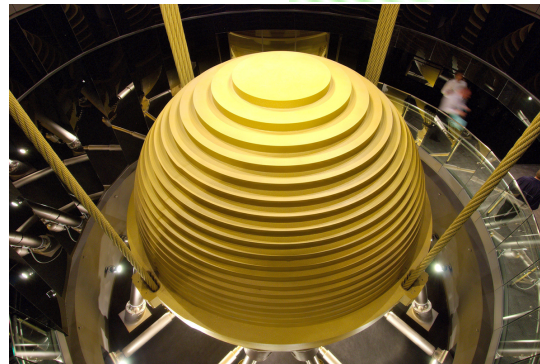
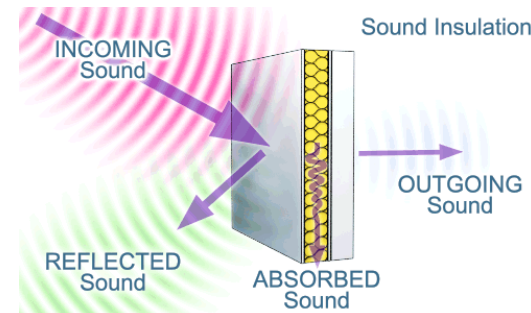
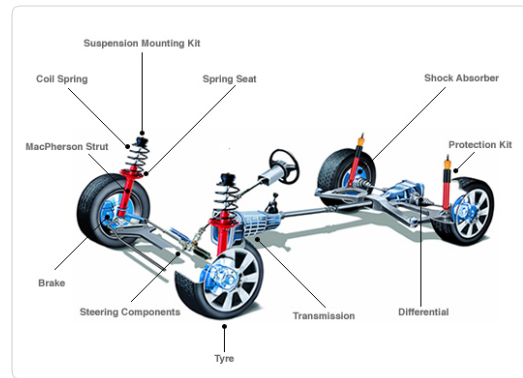
So far we have seen examples of **first-order DEs**, or equations with first derivatives of some unknown function.

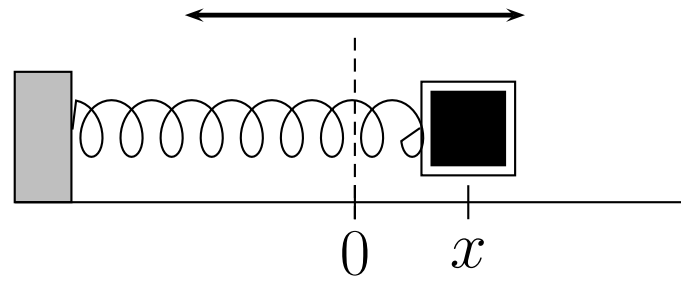
From here on in the course, we will study differential equations with **second or higher derivatives**.

One classic source of differential equations of this type comes from analyzing the forces on a block at the end of a spring.



While the mathematics behind this simple system will be very interesting in their own right, we should also note at the outset that the simple spring/mass model can be applied to a wide variety of not-so-obviously related real-world problems.





In this system, how would you describe  $x$  in words?

**Problem.** Draw a free-body diagram for the mass. Indicate the magnitude of the forces, assuming

- the mass of the block is  $m$  kg, and
- the spring constant (in  $N/m$ ) is given by the constant  $k$ .

Let us work with our intuition about this system before beginning the mathematics.

If the spring is very stiff, is  $k$  large or small?

**Definition.** Period: the length of time to complete one full cycle/oscillation.

If we increase the stiffness of the spring, do you expect the *period* of the oscillations to increase or decrease? Why?

If we increase the mass, do you expect the *period* of the oscillations to increase or decrease? Why?

If we know  $k$  and  $m$ , and assume that friction is negligible, should we be able to determine the exact period of the oscillations?

From the work so far, can we easily find the formula for the period?

The spring system is an excellent introduction to higher-order differential equations because

- we all have an intuition about how it *should* work physically,
- the mathematics and physics are simple, and
- there's no obvious way to predict critical features (e.g. the period) from the given information.

We clearly need some new tools!

Use Newton's second law,  $F = ma$ , to construct an equation involving the position  $x(t)$ .

What order of differential equation does  $F = ma$  produce for this spring/mass system?

To simplify matters temporarily, let us assume that both  $k = 1$  N/m and  $m = 1$  kg. Rewrite the previous differential equation.

This differential equation invites us to find a function  $x(t)$  whose second derivative is its own negative. What function(s) would satisfy that?

Having found two (and more) solutions to the differential equation for the spring/mass system, we now need to know how we can reliably **unify** those solutions. We will do so by looking at the general case of second-order linear differential equations.

A **linear second-order equation** can be expressed in the form  $A(x)y'' + B(x)y' + C(x)y = F(x)$ . Assuming that  $A(x) \neq 0$ , we can rewrite the equation in the **standard form**:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x).$$

Compare this form with the differential equation for the spring/mass system.

Open question: what is the form of the **general solution** to a linear second-order linear differential equations?

**Lemma.** *If both  $y_1$  and  $y_2$  are solutions to the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0,$$

*then any linear combination  $C_1y_1 + C_2y_2$  is also a solution. In other words, the solutions form a vector space, with  $y_1$  and  $y_2$  being a basis.*<sup>1</sup>

Prove this.

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<sup>1</sup>For key ideas from linear algebra that are useful for differential equations, see the supplementary materials.

$$y'' + p(x)y' + q(x)y = 0$$

**Problem.** Verify that  $y_1(x) = e^{2x} \cos(3x)$  and  $y_2(x) = e^{2x} \sin(3x)$  are solutions to  $y'' - 4y' + 13y = 0$ .

$$y'' - 4y' + 13y = 0$$

After verifying, construct the general solution to the DE.

$$y_1(x) = e^{2x} \cos(3x), \quad y_2(x) = e^{2x} \sin(3x)$$
$$y'' - 4y' + 13y = 0.$$

Find a particular solution that satisfies the initial conditions  $y(0) = 2$  and  $y'(0) = -5$ .

$$y_1(x) = e^{2x} \cos(3x), \quad y_2(x) = e^{2x} \sin(3x)$$
$$y'' - 4y' + 13y = 0.$$

One of the important ideas from linear algebra is that two solutions can be combined to make the general solution **only if the two solutions are linearly independent**.

**Definition.** Two functions  $y_1(x)$  and  $y_2(x)$  are linearly independent if and only iff the values  $a = 0$ ,  $b = 0$  are the only solutions to  $ay_1(x) + by_2(x) = 0$ .

**Problem.** Determine whether  $\sin(t)$  and  $\cos(t - \pi/2)$  are linearly independent.

For two differentiable function  $y_1$  and  $y_2$ , the ***Wronskian*** is

$$W[y_1, y_2] := \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

**Lemma.** *If the Wronskian is nonzero at some point, then  $y_1$  and  $y_2$  are linearly independent.*

Prove this lemma.

**Problem.** Use the Wronskian to show that  $x$  and  $e^x$  are linearly independent functions.

**Problem.** Given that  $y_1 = e^{-2t}$  and  $y_2 = e^{-3t}$  are both solutions to

$$y'' + 5y' + 6y = 0,$$

find the general solution.

$y'' + 5y' + 6y = 0$ , with  $y_1 = e^{-2t}$  and  $y_2 = e^{-3t}$  as solutions.

**Problem.** Show that  $\sqrt{x}$  and  $\frac{1}{x}$  span the solution space to  $2x^2y'' + 3xy' - y = 0$  when  $x > 0$ .

$$2x^2y'' + 3xy' - y = 0$$

Given that  $y_1 = \sqrt{x}$  and  $y_2 = \frac{1}{x}$  are both solutions to

$$2x^2y'' + 3xy' - y = 0 \text{ when } x > 0,$$

find the general solution.

## Homogeneous Equations with Constant Coefficients

So we know that if we find two solutions to a second order linear DE, we can always build the general solution. But now, how does we find those two solutions?

It turns out to be difficult with the general linear DE case, so we will start with the

- homogeneous (RHS of the DE equals 0 in standard form)
- linear DE ( $y$  and its derivatives are linear)
- with constant coefficients (all the coefficients will be constants, not functions of  $x$ )

How does our spring/mass DE relate to these restrictions?

Classify the following DEs based on the terms *homogeneous*, *linear* and *constant coefficients*

$$x^2y'' + xy' + y = 10$$

$$100y'' + y = 4x^3$$

$$(y'')^2 + y = 4e^x$$

$$4y'' - 10y' + y = 0$$

The generic 2nd order DE with constant coefficients is of the form

$$ay'' + by' + cy = 0$$

where  $a, b, c \in \mathbb{R}$ .

To find a solution, we will make an informed guess at the solution form, and see what happens.

**Problem.** Suppose that  $y = e^{rx}$  for some  $r \in \mathbb{R}$  is a solution. What happens when we sub that proposed solution into the differential equation?

Starting with

$$ay'' + by' + cy = 0$$

and the assumed solution form  $y = e^{rx}$ , we obtain the ***characteristic equation***:

$$ar^2 + br + c = 0$$

If  $r$  is a root of the characteristic equation, then  $y = e^{rx}$  is a solution to the differential equation.

**Problem.** Find the general solution to  $y'' + 5y' - 6y = 0$

**Proposition.** *If the characteristic equation has exactly two distinct roots  $r_1$  and  $r_2$ , then the general solution is  $C_1e^{r_1x} + C_2e^{r_2x}$ .*

Prove this.

**Problem.** Solve  $x'' + 2x' - x = 0$ ;  $x(0) = 0$ ,  $x'(0) = -1$ .

$$x'' + 2x' - x = 0; x(0) = 0, x'(0) = -1.$$

**Problem.** Solve  $y'' + 4y' + 4y = 0$

## Reduction of Order

How can we explain the second solution from our previous example? Let  $r$  be a root of the characteristic equation of  $ay'' + by' + cy = 0$  so that  $e^{rx}$  is a solution.

One hypothesis for a second solution would be “a function like/related to the first solution”, and one way to design such a function would be multiply our known solution by an unknown function,  $u(x)$ :

$$y_2 = u(x)e^{rx}$$

If this new form is in fact a valid solution, we can verify that by subbing it into the original DE.

$ay'' + by' + cy = 0$ ;  $y_1 = e^{rx}$  is a solution.

**Problem.** Show that  $y_2 = u(x)e^{rx}$  is also a solution, for certain choices of  $u(x)$ .

**Proposition.** *If the characteristic equation has exactly one double root  $r$ , then the general solution is  $C_1e^{rx} + C_2xe^{rx}$ .*

Prove this.

**Problem.** Solve  $y'' + y = 0$ .

**Remark.** Using power series, we have

$$\begin{aligned} e^{\sqrt{-1}\theta} &= 1 + (\sqrt{-1}\theta) + \frac{(\sqrt{-1}\theta)^2}{2!} + \frac{(\sqrt{-1}\theta)^3}{3!} + \dots + \frac{(\sqrt{-1}\theta)^n}{n!} + \dots \\ &= 1 + \sqrt{-1}\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}\sqrt{-1} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}\sqrt{-1} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + \sqrt{-1}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\ &= \cos(\theta) + \sqrt{-1}\sin(\theta). \end{aligned}$$

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**Proposition.** *If the characteristic equation has exactly two conjugate roots  $a \pm b\sqrt{-1}$ , then the general solution is  $C_1 e^{ax} \cos(bx) + C_2 e^{ax} \sin(bx)$ .*

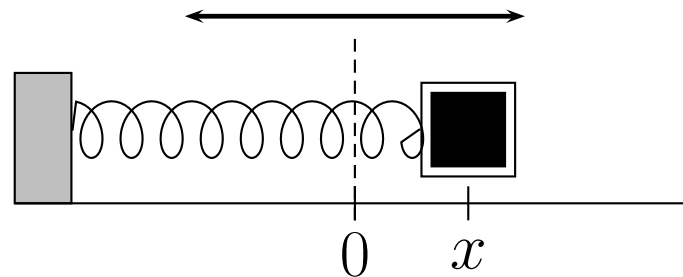
## **2nd Order Homogeneous DEs - General Case**

How does one solve homogeneous linear equations?

**Problem.** Solve  $16y'' - 8y' + 145y = 0$ ,  $y(0) = -2$ ,  $y'(0) = 1$ .

$$16y'' - 8y' + 145y = 0, y(0) = -2, y'(0) = 1.$$

# Spring/Mass System Revisited



The position over time of the mass in this system is dictated by Newton's Second Law,  $F = ma$ . With the spring force given by  $-kx$ ,  $a = x''$ , the system must satisfy

$$mx'' = -kx$$

or

$$x'' + \frac{k}{m}x = 0$$

$$x'' + \frac{k}{m}x = 0$$

**Problem.** Find the general solution to this differential equation.

Use the general solution to predict the period of the oscillations, if you were given the spring constant  $k$  and the mass  $m$ .