Week #9 : DEs with Non-Constant Coefficients, Laplace Resonance

Goals:

- Solving DEs with Non-Constant Coefficients
- Resonance with Laplace
- Laplace with Periodic Functions

Solving Equations with Non-Constant Coefficients

Problem. Consider the IVP

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

What techniques from the course could we use to solve this equation?

None!

$y_c \to$ $y = e^{rt}$ based on $y, y', y'' \ldots$

would end up cancelling out.

Laplace : $\mathcal{L}(t \cdot y') = ?$

So far... but maybe w/ some work

Proposition (Frequency differentiation). *If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order a, then for $s > a$ we have*

power of t

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\Big(\mathcal{L}\{f(t)\}(s)\Big).$$

second funct.

Problem. Sketch the proof of this relationship.

$\mathcal{L}(e^{at} f(t))$

$= F(s-a)$

for $n=1$, $\mathcal{L}(t \cdot f(t))$ eg $\mathcal{L}(t e^{2t})$

swap order $\mathcal{L}(t \sin(t))$

$$\frac{d}{ds}\mathcal{L}(f(t)) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt$$

def'n of $\mathcal{L}$

Repeat / use induction for $t^2, t^3 \ldots$

$-\mathcal{L}\{t f(t)\}$

$$= \int_0^\infty \left[\frac{d}{ds} f(t)\, e^{-st}\right] dt$$

const w/s

$$= \int_0^\infty f(t)\, e^{-st}\cdot(-t)\, dt = -\int_0^\infty t f(t)\, e^{-st}\, dt$$

Problem. Use the general transform

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\Big(\mathcal{L}\{f(t)\}(s)\Big)$$

to compute

$$\mathcal{L}\{t^{1} f(t))\} = -\frac{d}{ds}\,\mathcal{L}\underbrace{(f(t))}_{\text{s's in it}}$$

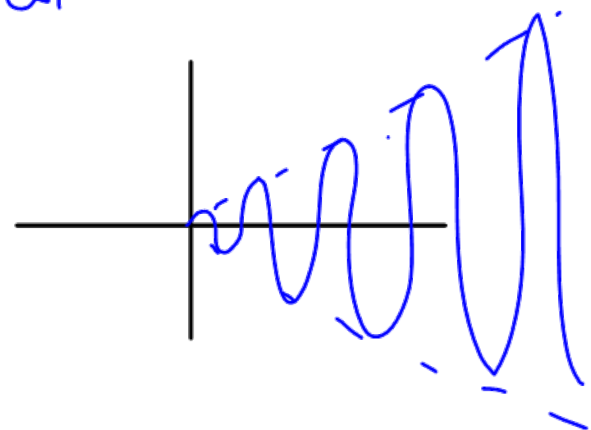
$$\mathcal{L}\{tf(t)\} = -1\frac{d}{ds}\Big(\mathcal{L}\{f(t)\}\Big)$$

Problem. Compute

$$\mathcal{L}\{t\sin(kt)\}$$

amplitude

appears in spring/mass w/ resonance



$$= -1\left(\frac{d}{ds}\left(\mathcal{L}(\sin(kt))\right)\right)$$

$\mathcal{L}(\sin(kt))$

$$= -1\ \frac{d}{ds}\left(\frac{k}{s^2+k^2}\right)$$

apply $\frac{d}{ds}$

$$= -1 \quad k(-1)(s^2+k^2)^{-2}\cdot 2s$$

$$= \frac{2ks}{(s^2+k^2)^2}$$

IVPs with Non-Constant Coefficients - Example 1

Problem. Use

$$\mathcal{L}\{tf(t)\} = -1\frac{d}{ds}\Big(\mathcal{L}\{f(t)\}\Big)$$

to help solve

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

non-constant

$\mathcal{L}$ of DE

$$\Big[s^2 Y(s) - s \cdot \underbrace{0}_{y(0)} - \underbrace{0}_{y'(0)}\Big] + 2\mathcal{L}(ty') - 4Y(s) = \frac{1}{s}$$

$$= 2(-1)\frac{d}{ds}\Big(\mathcal{L}(y')\Big)$$

$$= -2\frac{d}{ds}\big(sY(s) - 0\big)$$ prod rule

$$-2\big[Y(s) + sY'(s)\big]$$

$$s^2 Y(s) - 2\big[Y(s) + s\,Y'(s)\big] - 4Y(s) = \frac{1}{s}$$

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

$$s^2 Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}$$

$$\frac{[s^2 - 2 - 4]}{-2s} Y(s) \frac{-2s}{-2s} Y'(s) = \frac{1}{s}\frac{1}{-2s}$$

New!

$$① \quad Y'(s) + \left(\frac{-s}{2} + \frac{3}{s}\right) Y(s) = \frac{1}{-2s^2}$$

$\leftarrow$ 1st order linear DE in $Y(s)$ std form

Integrating factors:

$$u(s) = e^{\int \left(-s/2 + 3/s\right) ds}$$

$$= e^{-s^2/4 + 3\ln(s)} = e^{-s^2/4} \cdot e^{3\ln(s)} = s^3 e^{-s^2/4}$$

$=s$

Mult both sides of std form ①

$$① \quad s^3 e^{-s^2/4} Y'(s) + \left(\frac{-s}{2} + \frac{3}{s}\right) s^3 e^{-s^2/4} Y(s) = \frac{-1}{2s^2} s^3 e^{-s^2/4}$$

d/ds

d/ds

prod rule

$$\frac{d}{ds}\left(s^3 e^{-s^2/4} \cdot Y(s)\right) = \frac{-1}{2\cancel{s^2}} s^{\cancel{3}} e^{-s^2/4}$$

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

$$\cancel{\int} \cancel{\frac{d}{ds}}\left(s^3 e^{-s^2/4}\, Y(s)\right) ds = \int -\frac{1}{2}\, s\, e^{-s^2/4}\, ds$$

$$\frac{s^3 e^{-s^2/4}\, Y(s)}{s^3 e^{-s^2/4}} = -\frac{1}{2}\, e^{-s^2/4}(-2) + C = \frac{e^{-s^2/4} + C}{s^3 e^{-s^2/4}}$$

$$Y(s) = \frac{1}{s^3} + \frac{C\, e^{s^2/4}}{s^3 \cancel{e^{-s^2/4}}}$$

← not exp'l order

Need $C = 0$

so $Y(s) = \frac{1}{2}\frac{2}{s^3}$

$\Rightarrow$ $$\boxed{y(t) = \frac{1}{2} t^2}$$

$$y'' + 2ty' - 4y = 1 \text{ with } y(0) = y'(0) = 0$$

Problem. Verify that your solution is correct.

$$y = \frac{1}{2}t^2$$

to sub in, need $y' = \frac{2t}{2} = t$

$$y'' = 1$$

$$\begin{aligned} LHS &= 1 + 2t(t) - 4\left(\tfrac{1}{2}t^2\right) \\ &= 1 + 2t^2 - 2t^2 \\ &= 1 \quad = RHS \end{aligned}$$

Laplace with 1/t Multipliers - Frequency Integration

A related property can be helpful when $\dfrac{1}{t}$ multipliers are present.

Proposition (Frequency Integration)**.** *Let $f(t)$ be piecewise continuous on $[0,\infty)$, of exponential order a, and $\displaystyle\lim_{t\to 0^+}\frac{f(t)}{t}$ is finite. If $F(s) := \mathcal{L}\{f(t)\}(s)$, then we have*

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_s^\infty F(\sigma)\,d\sigma \text{ for } s > a$$

$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)$

t mult → deriv in s

t divisor → integ'l in s

Problem. Compute $\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}$.

$$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$$

$$= \int_s^\infty F(\sigma)\, d\sigma$$

$$= \int_s^\infty \frac{1}{\sigma^2+1}\, d\sigma$$

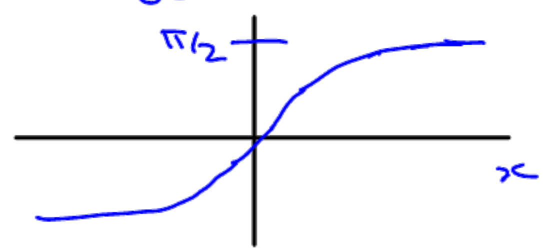
$$= \lim_{b\to\infty} \arctan(\sigma)\Big|_s^b$$

$$= \lim_{b\to\infty} \left(\arctan(b) - \arctan(s)\right)$$

$$= \pi/2 - \arctan(s)$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$$

$\arctan(x)$



IVPs with Non-Constant Coefficients - Example 2

Problem. Solve $ty'' + 2y' + ty = 0$, $y(0) = 1$, and $y(\pi) = 0$ boundary value

non-constant

still Linear

$\mathcal{L}$ of DE

$$\mathcal{L}\{ty''\} + 2[sY(s) - y(0)] + \mathcal{L}\{ty\} = 0$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}\left[s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)}\right]$$

$y(0) = 1$, $y'(0) = a$

$$\mathcal{L}\{ty\} = -\frac{d}{ds}\mathcal{L}\{y\} = -Y'(s)$$

$$-\left[2sY(s) + s^2Y'(s) - 1 + 0\right] + 2\left[sY(s) - 1\right] - Y'(s) = 0$$

$$(-s^2-1)Y'(s) + (\cancel{-2s + 2s})Y(s) = -1 + 2$$

$$Y'(s) = \frac{-1}{(s^2+1)}$$

Hint. This problem doesn't immediately use our new $1/t$ integration theorem, but wait for it...

$$ty'' + 2y' + ty = 0,\ y(0) = 1, \text{ and } y(\pi) = 0$$

$$Y'(s) = \frac{-1}{s^2+1}$$

integrate both sides wrt s

$$Y(s) = \int \frac{-1}{s^2+1}\, ds$$

$$Y(s) = -\arctan(s) + C$$

use property

$$0 = \lim_{s\to\infty} \left(-\underbrace{\arctan(s)}_{=\,\pi/2} + C \right)$$

$Y(s) \to 0$ as $s \to \infty$

$$C = \pi/2$$

so $\boxed{Y(s) = -\arctan(s) + \pi/2}$

not in table!

previous work

$$y(t) = \frac{\sin(t)}{t} \rightarrow y(\pi) = \frac{\sin(\pi)}{\pi} = 0 \ ✓$$

$$\lim_{t\to 0} y(t) = 1 \ ✓$$

$$ty'' + 2y' + ty = 0, \ y(0) = 1, \text{ and } y(\pi) = 0$$

$$ty'' + 2y' + ty = 0,\ y(0) = 1, \text{ and } y(\pi) = 0$$

Problem. Confirm that your solution is correct.

$$y = \frac{\sin(t)}{t}$$

To sub in, need $y' = \dfrac{\cos(t)\,t - \sin(t)}{t^2}$

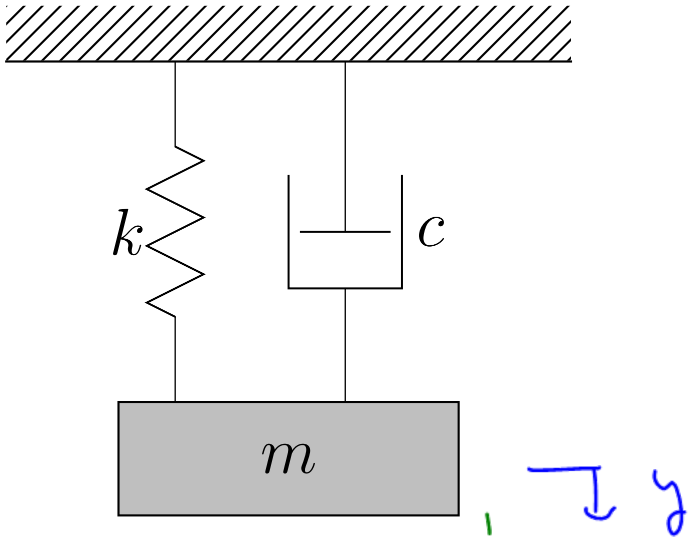
and $y'' = \dfrac{(-\sin(t)\cdot t + \cancel{\cos(t)} - \cancel{\cos(t)})\,t^2 - (\cos(t)\,t - \sin(t))\,2t}{t^4}$

$$= \frac{\sin(t)[-t^3 + 2t] + \cos(t)(-2t^2)}{t^4}$$

LHS $\left[\underbrace{\dfrac{\sin(t)[-t^3+2t] + \cos(t)(-2t^2)}{t^3}}_{ty''}\right] + 2\left[\dfrac{\cos(t)\cdot t - \sin(t)}{t^2}\right] + \underbrace{\sin(t)}_{ty}$

$$= -\cancel{\sin(t)} + 2\cancel{\frac{\sin(t)}{t^2}} - 2\cancel{\frac{\sin(t)}{t^2}} + \cancel{\sin(t)} + \cancel{\frac{(-2)\cos(t)}{t}} + 2\cancel{\frac{\cos(t)}{t}} = 0$$

Spring/Mass System Resonance With Laplace



Problem.

Write out the DE for the position of the mass, given $F_{\text{ext}} = F_0 \sin(\omega t)$.

$$ma = \sum F$$

$$my'' = F_{sp} + F_d + F_{ext}$$

sinusoidal/periodic

$$my'' = -ky + (-cy') + F_0 \sin(\omega t)$$

ext force

$$my'' + cy' + ky = F_0 \sin(\omega t)$$

$$my'' + cy' + ky = F_0 \sin(\omega t)$$

F_{ext}

Problem. If we set $m = 1$, $c = 0$ and $\omega = \sqrt{k}$ (or $k = \omega^2$), what would this mean for the physical system?

$$y'' + ky = F_0 \sin(\sqrt{k}\, t)$$

Natural freq (frequency of unforced system)

$y_c \Rightarrow \quad r^2 + k = 0$

$$r^2 = -k$$

$$r = \pm\sqrt{k}\,\sqrt{-1}$$

$$y_c = c_1 \cos(\sqrt{k}\,t) + c_2 \sin(\sqrt{k}\,t)$$

oscillations @ $\sqrt{k}$ rad/s

F_{ext} is in sync w/ natural oscillations

$$y'' + \omega^2 y = F_0 \sin(\omega t) \text{ - no damping, matching frequencies}$$

Problem. Predict the position of the mass over time, given that it starts at equilibrium; use Laplace transforms.

$\mathcal{L}$ of DE

$$\left[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} \right] + \omega^2 Y(s) = F_0 \frac{\omega}{s^2+\omega^2}$$

$$(s^2+\omega^2) Y(s) = F_0 \frac{\omega}{s^2+\omega^2}$$

$$Y(s) = F_0 \frac{\omega}{(s^2+\omega^2)^2}$$

Not in table

Recall: sol'n involve

$t\sin(\omega t)$, $t\cos(\omega t)$

$y'' + \omega^2 y = F_0 \sin(\omega t)$ - no damping, matching frequencies

$\mathcal{L}\{t \sin(\omega t)\}$

$= -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$

$= -(-1)(s^2+\omega^2)^{-2} \cdot \omega \cdot 2s$

$= \frac{2s\omega}{(s^2+\omega^2)^2}$

left out F_0

$\mathcal{L}\{t \cos(\omega t)\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$

$= -\left[\frac{(s^2+\omega^2) - s(2s)}{(s^2+\omega^2)^2}\right]$

$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$Y(s) = \frac{-\omega^2}{(s^2+\omega^2)^2} \cdot \frac{-1}{\omega} = -\frac{1}{\omega}\left[\frac{-\omega^2 + s^2 - s^2}{(s^2+\omega^2)^2}\right] \cdot F_0$

$= -\frac{1}{\omega}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2} - \frac{s^2 + \omega^2 - \omega^2}{(s^2+\omega^2)^2}\right] \cdot F_0$

$Y(s) = -\frac{1}{\omega}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2} - \frac{1}{s^2+\omega^2} + \frac{\omega^2}{(s^2+\omega^2)^2}\right] = \left[\sim - \sim\right] - \overbrace{\frac{\omega}{(s^2+\omega^2)^2}}^{Y(s)}$

$$y'' + \omega^2 y = F_0 \sin(\omega t) \text{ - no damping, matching frequencies}$$

$$\cancel{2}\,Y(s) = \frac{-1}{2\omega}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2} - \frac{1}{\omega}\,\frac{\cancel{1}\,\omega}{s^2+\omega^2}\right]$$

so $\mathcal{L}^{-1}(Y(s))$

$$= \frac{-1}{2\omega}\left[t\cos(\omega t) - \frac{1}{\omega}\sin(\omega t)\right]$$

$$\boxed{y(t) = \frac{1}{2\omega^2}\sin(\omega t) - \frac{1}{2\omega}\,t\cos(\omega t)}$$

resonance $\Longleftrightarrow$ amplitude always growing

Spring/Mass System with Square-Wave Forcing

Problem. For a spring/mass system exhibiting resonance, using $F_{\text{ext}} = F_0 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$, what element in the external force seems the most relevant to causing resonance?

mostly the matching freq's.

What other F_{ext} functions might produce the same ever-growing oscillation amplitude?

periodic functions w/ same period as natural/unforced oscillations.

Problem. Find the natural **period** of the spring/mass system defined by

$$y'' + \frac{\pi^2}{4} y = 0$$

$\pi^2 \approx 10$

$y_c \Rightarrow \quad r^2 + \frac{\pi^2}{4} = 0 \qquad r = \pm\sqrt{\frac{\pi^2}{4} \cdot -1}$

$r = \pm \frac{\pi}{2} \cdot \sqrt{-1}$

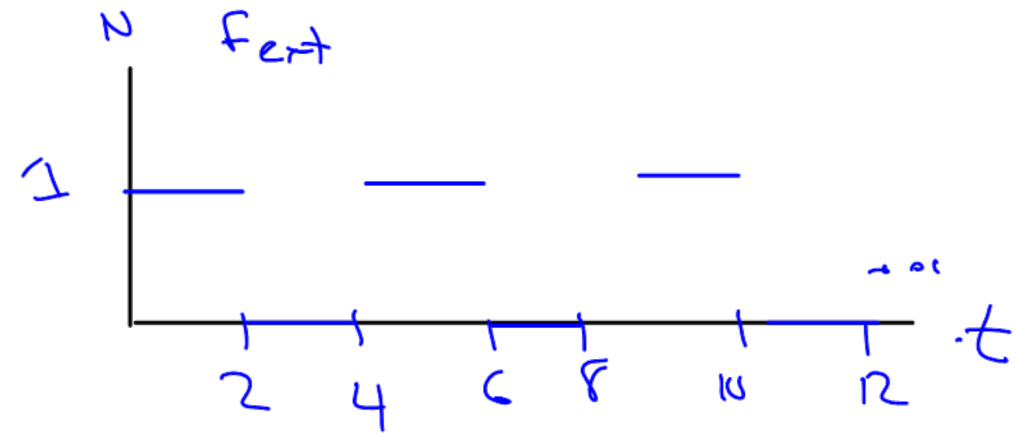
$y_c = c_1 \cos\left(\frac{\pi}{2} t\right) + c_2 \sin\left(\frac{\pi}{2} t\right)$

freq (rad/s)

$\text{Period} = \frac{2\pi}{\pi/2} = 4$ seconds

$$y'' + \frac{\pi^2}{4} y = F_{\text{ext}}$$

Problem. Write an F_{ext} function that would push at 1 N for half of a cycle, then nothing for the rest of the cycle, push for a half cycle, then off again, etc.



Problem. Write F_{ext} using step functions.

$$F_{ext} = (u_0 - u_2) + (u_4 - u_6) + (u_8 - u_{10}) + \dots$$

Problem. Find $\mathcal{L}\{F_{\text{ext}}\}$.

$$\mathcal{L}\{\ u_0 - u_2 + u_4 - u_6 + u_8 - u_{10} \ldots \}$$

$$= \frac{e^{-0s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-6s}}{6} + \ldots$$

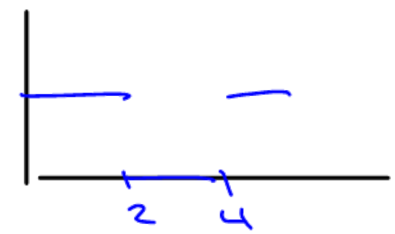
Problem. Predict the motion of the spring/mass system

$$y'' + \frac{\pi^2}{4}y = F_0\, f\text{sq}(t),\;\; f\text{sq}(t) = \begin{cases} 1 & 0 \le t < 2 \\ 0 & 2 \le t < 4 \\ 1 & 4 \le t < 6 \\ 0 & 6 \le t < 8 \\ \text{etc.} & \end{cases}$$

↳ $\mathcal{L}$ of DE

$$\left[s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0}\right] + \frac{\pi^2}{4} Y(s)$$

$$= \mathcal{L}(F_0\, f_{sq})$$



$$\left(s^2 + \pi^2/4\right) Y(s) = F_0\left(\frac{e^{-0s}}{s} - \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-6s}}{s} + \dots\right)$$

$$Y(s) = \frac{1}{s^2 + \pi^2/4}\, F_0 \left(\frac{e^{-0s}}{s} - \frac{e^{-2s}}{s} + \dots -\right)$$

$$= F_0\left[e^{-0s} - e^{-2s} + e^{-4s} - e^{-6s} + \dots\right]\left(\frac{1}{s(s^2 + \pi^2/4)}\right)$$

$$y'' + \frac{\pi^2}{4} y = F_0\ f\text{sq}(t)$$

$$\frac{1}{s(s^2+\pi^2/4)} = \frac{A}{s} + \frac{Bs+C}{s^2+\pi^2/4}$$

Same freq as the unforced osc'ns

$$= \frac{4}{\pi^2}\cdot\frac{1}{s} - \frac{4}{\pi^2}\,\frac{s}{s^2+\pi^2/4}$$

$$Y(s) = F_0\left[e^{-0s} - e^{-2s} + e^{-4s} - e^{-6s} + \cdots\right]\cdot\left[\frac{4}{\pi^2}\,\frac{1}{s} - \frac{4}{\pi^2}\,\frac{s}{s^2+\pi^2/4}\right]$$

$\mathcal{L}^{-1}$ of $Y(s)$

$$1 - \left(-\cos\left(\tfrac{\pi}{2}t\right)\right)$$

$$= F_0\frac{4}{\pi^2}\left[1-\cos\left(\tfrac{\pi}{2}t\right) - u_2\left(1-\cos\left(\tfrac{\pi}{2}(t-2)\right)\right) + u_4\left(1-\cos\left(\tfrac{\pi}{2}(t-4)\right)\right) - \cdots\right.$$

$$1-\cos(\pi/2\, t)$$

$$= F_0\frac{4}{\pi^2}\begin{cases} 1-\cos(\pi/2\, t) & 0\le t\le 2 \\ \cancel{1}-\cos(\pi/2\, t) - (\cancel{1}+\cos(\pi/2\, t)) = -2\cos(\pi/2\, t) & 2\le t\le 4 \\ 1-3\cos(\pi/2\, t) & 4\le t\le 6 \\ -4\cos(\pi/2\, t) & 6\le t\le 8\end{cases}$$

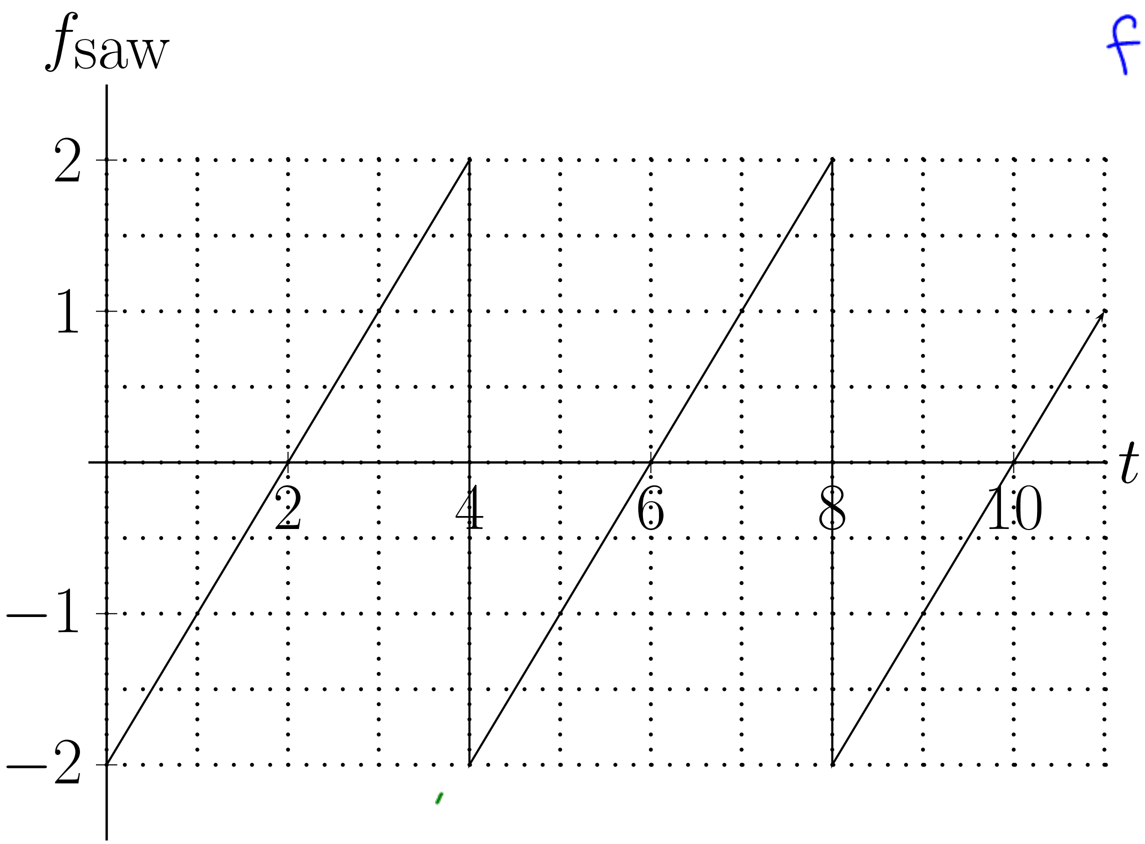
linearly growing amplitude of oscillation

$$y'' + \frac{\pi^2}{4} y = F_0 \, f_{\mathrm{sq}}(t)$$

Spring/Mass System with Sawtooth Forcing

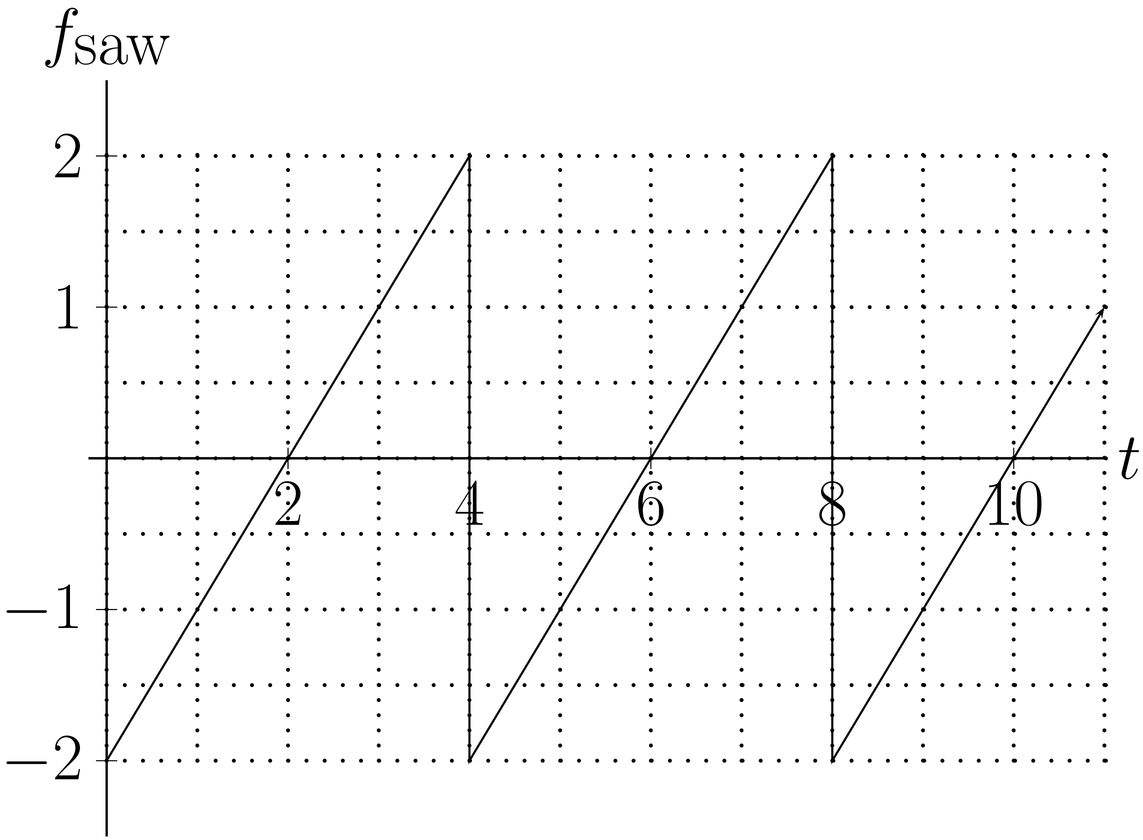
We will now study the effect on the spring/mass system using an alternative periodic forcing function, the *sawtooth* function.

Problem. The start of the graph of f_{saw} is shown below. Write out a formula for this periodic function.



$$f(t) = \begin{cases} t-2 & 0 \le t \le 4 \\ (t-4)-2 & 4 \le t \le 8 \\ (t-8)-2 & 8 \le t \le 12 \end{cases}$$

$u = t-6$



Problem. Write f_{saw} using step functions.

$$f_{ext} = (t-2)(u_0 - u_4) + [(t-4)-2](u_4 - u_8)$$
$$+[(t-8)-2](u_8 - u_{12}) + \sim$$

Problem. Find $\mathcal{L}\{f_{\text{saw}}\}$.

$$f_{saw} = (t-2)(u_0 - u_4) + [(t-4)-2](u_4 - u_8)$$
$$+[(t-8)-2](u_8 - u_{12}) + \cdots$$

$$\mathcal{L}(u_a \cdot f(t))$$
$$= e^{-as}\mathcal{L}(f(t+a))$$

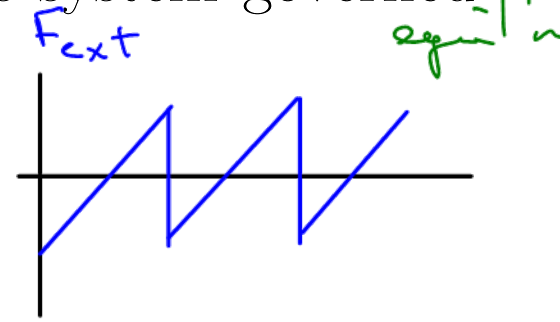
$$f_{saw} = (t-2) \quad + u_4(-\cancel{t} + \cancel{2} + \cancel{t} - 4 - \cancel{2})$$
$$+ u_8(-\cancel{t} + \underbrace{4 + \cancel{2} + \cancel{t} - 8}_{-4} - \cancel{2})$$
$$+ u_{12}(\underbrace{\ldots}_{-4}\ldots)$$

$$= (t-2) + (u_4 + u_8 + u_{12} + u_{16} \cdots)[-4]$$

$$\mathcal{L}\{f_{saw}\} = \left(\frac{1}{s^2} - \frac{2}{s}\right) + \left(-\frac{4}{s}\right)\left[e^{-4s} + e^{-8s} + e^{-12s} + \cdots\right]$$

Problem. Predict the motion of the spring/mass system governed by

$$y'' + \frac{\pi^2}{4} y = \underline{F_0}\, f_{\text{saw}}(t)$$



$\mathcal{L}$ of DE

$$\left[s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)}\right] + \frac{\pi^2}{4} Y(s) = F_0\left(\left(\frac{1}{s^2} - \frac{2}{s}\right) + \left(\frac{-4}{s}\right)\left[e^{-4s} + e^{-8s} + e^{-12s} + \dots\right]\right)$$

$$\left(s^2 + \pi^2/4\right) Y(s) = F_0\left(\frac{1}{s^2} - \frac{2}{s} + \left(\frac{-4}{s}\right)\left[e^{-4s} + e^{-8s} + e^{-12s} + \dots\right]\right)$$

$$Y(s) = F_0\left(\frac{1}{s^2(s^2+\pi^2/4)} - 2\frac{1}{s(s^2+\pi^2/4)} + (-4)\frac{1}{s(s^2+\pi^2/4)}\left[e^{-4s} + e^{-8s} + e^{-12s} + \dots\right]\right)$$

$$\frac{1}{s^2(s^2+\pi^2/4)} = \left(\frac{4}{\pi^2}\frac{1}{s^2} - \frac{4}{\pi^2}\frac{1}{s^2+\pi^2/4}\right)$$

part frac

$$\frac{1}{s(s^2+\pi^2/4)} = \left(\frac{4}{\pi^2}\frac{1}{s} - \frac{4}{\pi^2}\frac{1}{s^2+\pi^2/4}\right)$$

$$y'' + \frac{\pi^2}{4}y = F_0 \; f_{\text{saw}}(t)$$

$$Y(s) = F_0 \; \frac{4}{\pi^2}\Bigg[\frac{1}{s^2} - \left(\frac{1}{s^2+\pi^2/4}\,\frac{\pi}{2}\right)\cdot\frac{2}{\pi}$$

$$-2\,\frac{1}{s} + 2\left(\frac{1}{s^2+\pi^2/4}\,\frac{\pi}{2}\right)\cdot\frac{2}{\pi}$$

$$-4\left(\frac{1}{s}\right)\left[e^{-4s} + e^{-8s} + e^{-12s} + \dots\right]$$

$$+4\left(\frac{1}{s^2+\pi^2/4}\,\frac{\pi}{2}\right)\frac{2}{\pi}\left[e^{-4s} + e^{-8s} + e^{-12s} + \dots\right]\Bigg]$$

$\mathcal{L}^{-1}$:

$$y(t) = F_0 \; \frac{4}{\pi^2}\Bigg(t - \frac{2}{\pi}\sin\left(\tfrac{\pi}{2}t\right) - 2 + 2\cdot\frac{2}{\pi}\sin\left(\tfrac{\pi}{2}t\right)$$

$$-4\left(u_4 + u_8 + u_{12} + \dots\right) \qquad \sin\left(\tfrac{\pi}{2}t\right)$$

$$+4\,\frac{2}{\pi}\Big(u_4 \cancel{\sin\left(\tfrac{\pi}{2}(t-4)\right)} \cancel{+} u_8 \sin\left(\tfrac{\pi}{2}(t-8)\right)$$

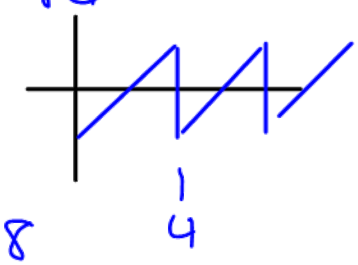
$$+ u_{12}\sin\left(\tfrac{\pi}{2}(t-12)\right) + \dots \Big)\Bigg)$$

$$y'' + \frac{\pi^2}{4}y = F_0 \; f_{\text{saw}}(t)$$

$$y(t) = F_0 \frac{4}{\pi^2}\Big[\, \underbrace{t - \frac{2}{\pi}\sin(\tfrac{\pi}{2}t) - 2 + 2\cdot\frac{2}{\pi}}_{g(t)} \sin(\tfrac{\pi}{2}t)$$

$$- 4\left(u_4 + u_8 + u_{12} + u_{16} + \cdots\right)$$

$$+ 4\cdot\frac{2}{\pi}\sin(\tfrac{\pi}{2}t)\left[u_4 + u_8 + u_{12} + u_{16} + \cdots\right]\Big]$$



$$y(t) = F_0 \frac{4}{\pi^2} \begin{cases} g(t) & 0 \le t \le 4 \\ g(t) - 4 + \boxed{\dfrac{8}{\pi}} \sin(\tfrac{\pi}{2}t) & 4 \le t \le 8 \\ g(t) - 8 + \boxed{\dfrac{16}{\pi}} \sin(\tfrac{\pi}{2}t) & 8 \le t \le 12 \\ g(t) - 12 + \boxed{\dfrac{24}{\pi}} \sin(\tfrac{\pi}{2}t) & 12 \le t \le 16 \end{cases}$$

increasing amplitudes

resonance

Consider an **undamped** spring/mass system.

Problem. If we use a periodic F_{ext} with the same frequency as the natural unforced oscillations, what do you expect to happen?

resonance $\Longleftrightarrow$ increasing amplitude of oscillations

Problem. If we use a periodic F_{ext} with *almost* the same frequency?

large (but not unbounded)

(beats)

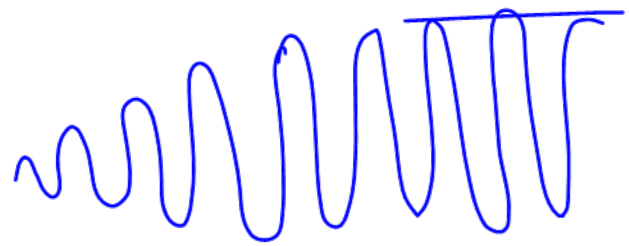
Problem. If we use a periodic F_{ext} with very different frequency?

generally small oscillations

Now consider an **under**damped spring/mass system. low damping.

Problem. If we use a periodic F_{ext} with the same frequency as the natural oscillations, what do you expect to happen?

effective practical
resonance



steady state oscillations w/ large amplitude.

Problem. If we use a periodic F_{ext} with *almost* the same frequency?

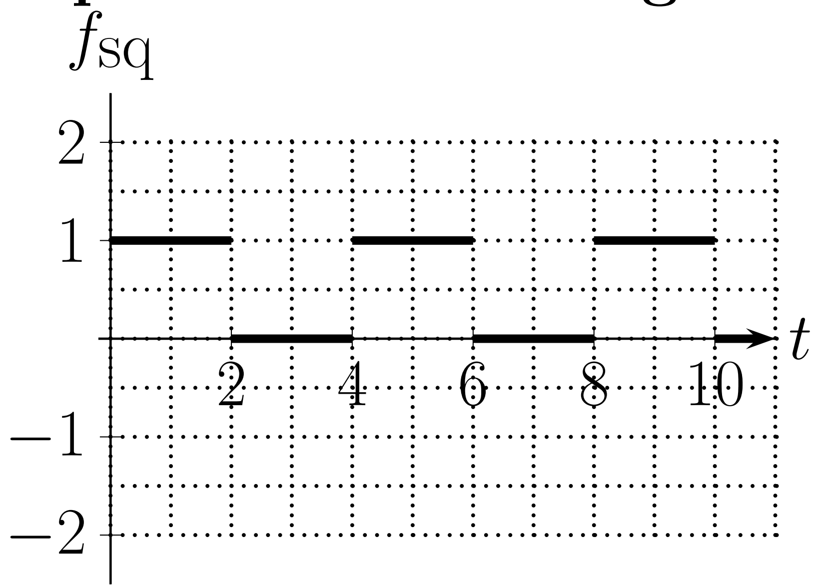
similar

Problem. If we use a periodic F_{ext} with very different frequency?

not much
(small amplitude oscillation)

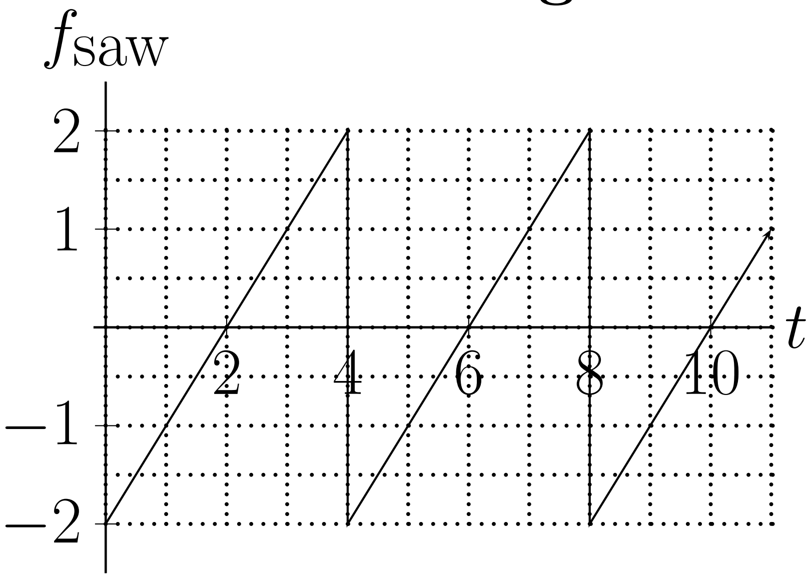
Spring/Mass System Demonstrations

Square-wave Forcing



$$y(t) = F_0 \frac{4}{\pi^2} \begin{cases} 1 - 1\sin\left(\frac{\pi}{2}t\right) & 0 \le t \le 2 \\ -2\sin\left(\frac{\pi}{2}t\right) & 2 \le t \le 4 \\ 1 - 3\sin\left(\frac{\pi}{2}t\right) & 4 \le t \le 6 \\ -4\sin\left(\frac{\pi}{2}t\right) & 6 \le t \le 8 \\ \vdots & \end{cases}$$

Sawtooth Forcing



$$y(t) = F_0 \frac{4}{\pi^2} \begin{cases} t - 2 + \frac{2}{\pi} \sin\left(\frac{\pi}{2} t\right) & 0 \le t \le 4 \\ t - 6 + \frac{2+8}{\pi} \sin\left(\frac{\pi}{2} t\right) & 4 \le t \le 8 \\ t - 10 + \frac{2+16}{\pi} \sin\left(\frac{\pi}{2} t\right) & 8 \le t \le 12 \\ t - 14 + \frac{2+24}{\pi} \sin\left(\frac{\pi}{2} t\right) & 12 \le t \le 16 \\ \vdots & \end{cases}$$