

Week #7 : Laplace - Step Functions, DE Solutions

Goals:

- Laplace Transform Theory
- Transforms of Piecewise Functions
- Solutions to Differential Equations

Existence of Laplace Transforms

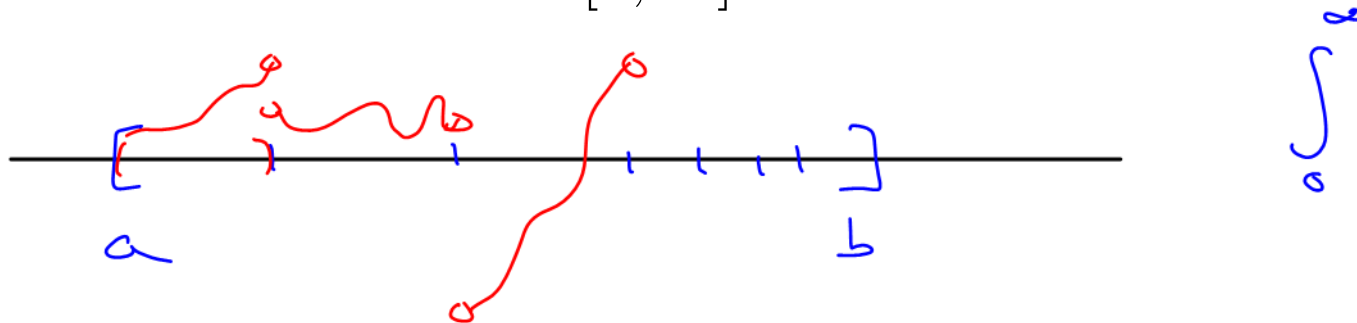
 $F(s)$

Before continuing our use of Laplace transforms for solving DEs, it is worth digressing through a quick investigation of which functions actually *have* a Laplace transform.

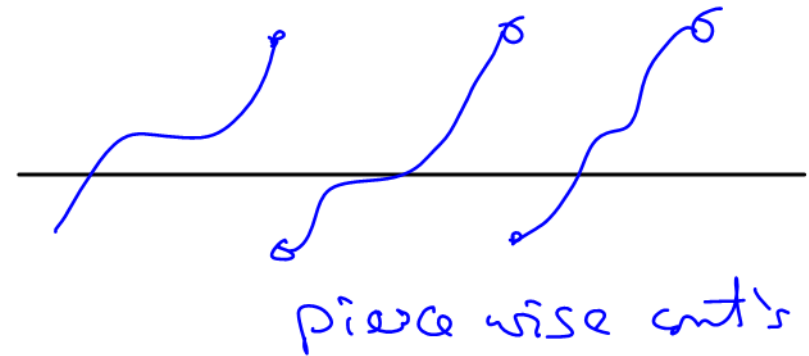
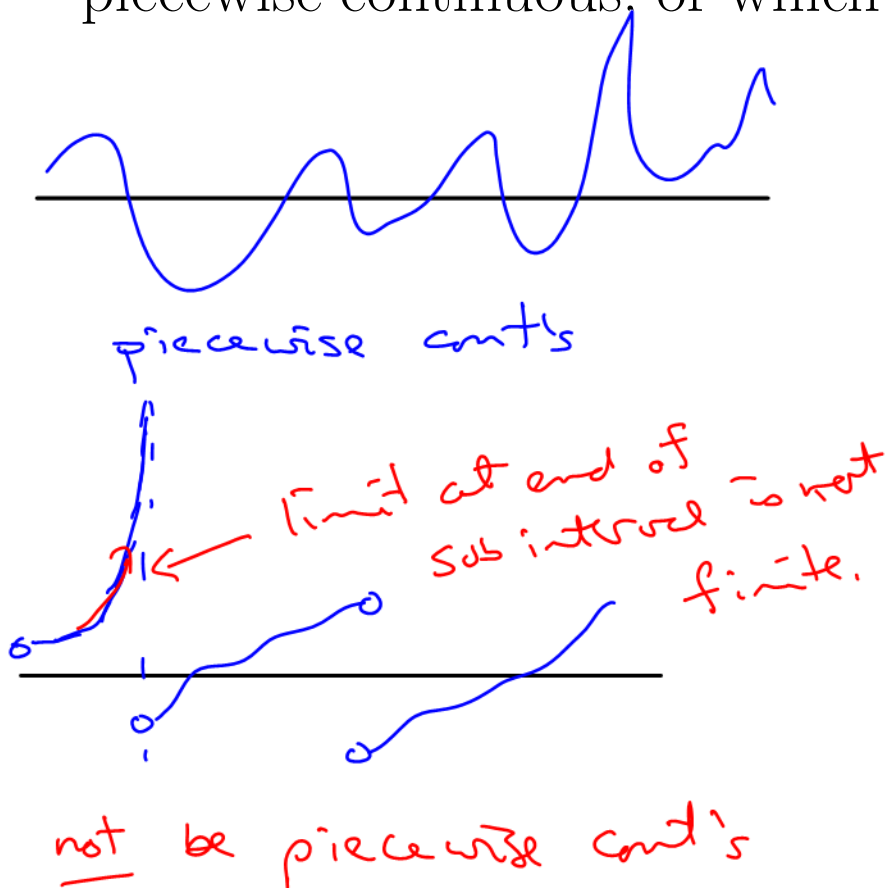
A function f is ***piecewise continuous*** on an interval $t \in [a, b]$ if the interval can be partitioned by a finite number of points $a = t_0 < t_1 < \dots < t_n = b$ such that

- f is continuous on each open subinterval (t_{i-1}, t_i) .
- f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is continuous on $[a, b]$ except for a finite number of jump discontinuities. A function is piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.



Problem. Draw examples of functions which are continuous and piecewise continuous, or which have different kinds of discontinuities.



One of the requirements for a function having a Laplace transform is that it be piecewise continuous. Classify the graphs above based on this criteria.

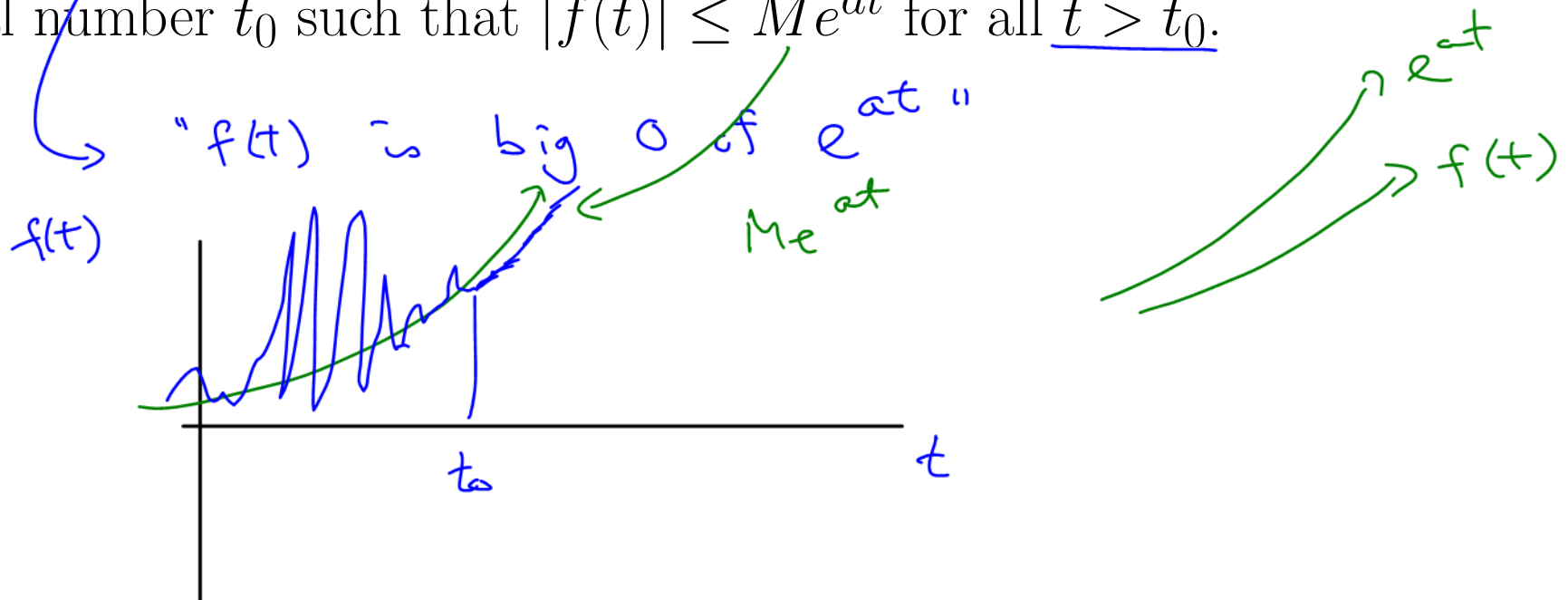
Another requirement of the Laplace transform is that the integral $\int_0^{\infty} e^{-st} f(t) dt$ converges for at least some values of s . To help determine this, we introduce a generally useful idea for comparing functions, "Big-O notation".

"OL"

"on the order of"

Big-O notation

We write $f(t) = O(e^{at})$ as $t \rightarrow \infty$ and say f is **of exponential order** a (as $t \rightarrow \infty$) if there exists a positive real number M and a real number t_0 such that $|f(t)| \leq Me^{at}$ for all $t > t_0$.



Lemma. Assume $\lim_{t \rightarrow \infty} \frac{|f(t)|}{e^{at}}$ exists. Then

$$\lim_{t \rightarrow \infty} \frac{|f(t)|}{e^{at}} < \infty \quad \text{finite}$$

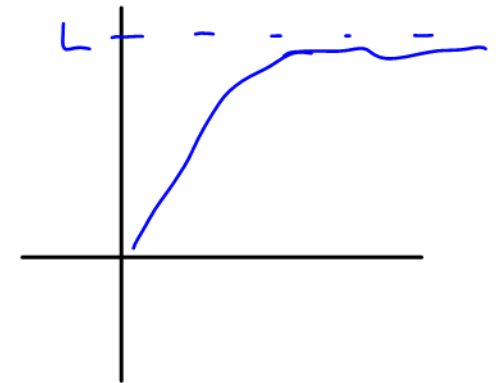
if and only if $f(t) = O(e^{at})$ as $t \rightarrow \infty$. \square

Problem. Show that bounded functions and polynomials are of exponential order a for all $a > 0$.

bounded f : $f \leq L$ for all t

$$\lim_{t \rightarrow \infty} \frac{|f(t)|}{e^{at}} \leq \lim_{t \rightarrow \infty} \frac{L}{e^{at}} = 0$$

finite



both bounded and poly'l
f's are of exp'l order.

poly's:

$$\lim_{t \rightarrow \infty} \frac{\text{poly}'l}{e^{at}} = 0$$

\curvearrowright
l'Hopital

Problem. Show that e^{t^2} does **not** have exponential order.

$$\lim_{t \rightarrow \infty} \frac{e^{t^2}}{e^{at}} = \lim_{t \rightarrow \infty} e^{(t^2 - at)} \rightarrow e^{\infty} \rightarrow \infty$$

e^{t^2} not of exp'l order

Problem. Are all the functions we have seen so far in our DE solutions of exponential order?

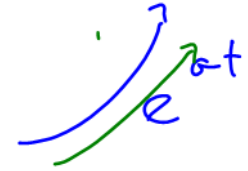
Yes!
 sin, cos: bounded ✓
 poly's ✓
 e^{at} ✓

The final reveal: what kinds of functions have Laplace transforms?

Proposition. *If f is*

- piecewise continuous on $[0, \infty)$ and
- of exponential order a ,

then the Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for $s > a$.



The proof is based the comparison test for improper integrals.

Laplace Transform of Piecewise Functions

In our earlier DE solution techniques, we could not directly solve non-homogeneous DEs that involved piecewise functions. Laplace transforms will give us a method for handling piecewise functions.

$$y'' + ay' + by = \underbrace{\begin{matrix} \sin(at) & \cos(at) \\ e^{bt} & \text{poly} \end{matrix}}$$

$$= \begin{cases} \sin(t) & 0 < t < 1 \\ \vdots \end{cases}$$

Problem. Use the definition to determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t \leq 5, & \rightarrow f_1 = 0 \text{ for } t \geq 5 \\ 0 & 5 < t \leq 10, \\ e^{4t} & 10 < t. & \rightarrow f_2 \end{cases}$$

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} \cdot f(t) dt$$

$$= \int_0^{\infty} e^{-st} f_1 dt + \int_0^{\infty} e^{-st} \cdot f_2 dt$$

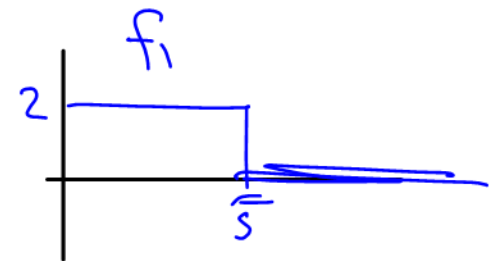
$$= \int_0^5 e^{-st} \cdot 2 dt$$

$f_1 = 2$ on $t=0 \dots 5$

$$+ \int_{10}^{\infty} e^{-st} \cdot e^{4t} dt$$

\leftarrow shortcut improper

$$= 2 \cdot \left. \frac{e^{-st}}{-s} \right|_0^5 + \left. \frac{e^{-st+4t}}{(-s+4)} \right|_{10}^{\infty}$$



$$= \frac{2}{-s} (e^{-st}) \Big|_0^5 + \frac{1}{-s+4} e^{-st+4t} \Big|_{t=10}^{\infty}$$

$$f(t) = \begin{cases} 2 & 0 < t \leq 5, \\ 0 & 5 < t \leq 10, \\ e^{4t} & 10 < t. \end{cases}$$

$$= \frac{2}{-s} (e^{-5s} - 1) + \frac{1}{-s+4} (0 - e^{-s \cdot 10 + 4 \cdot 10})$$

$$= \frac{2}{s} - \frac{2e^{-5s}}{s} + \frac{1}{s-4} e^{-10s} \cdot e^{40}$$

$\leftarrow e^{-s} \text{ are indicators of } f(t) \text{ being piecewise.}$

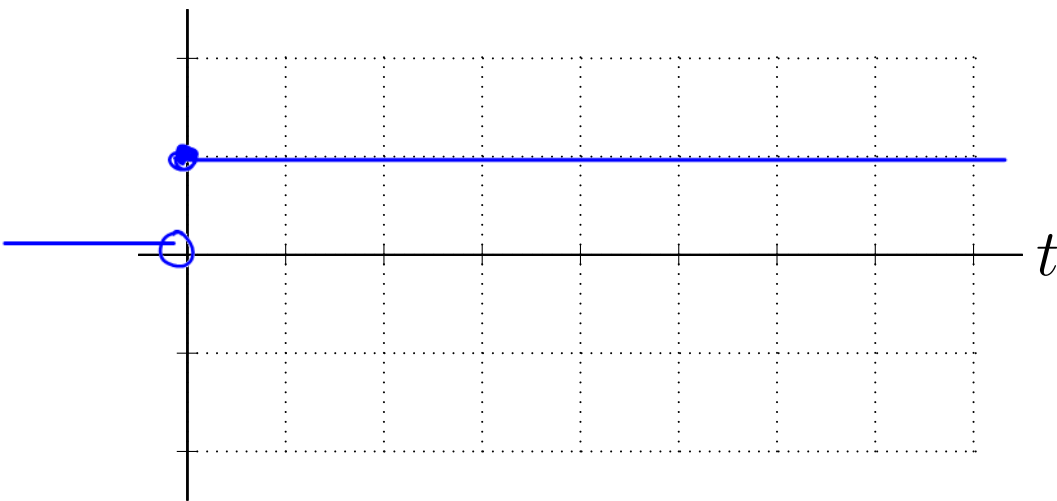
Similar to previous

We would like avoid having to use the Laplace definition integral if there is an easier alternative. A new notation tool will help to simplify the transform process.

The *Heaviside step function* or *unit step function* is defined

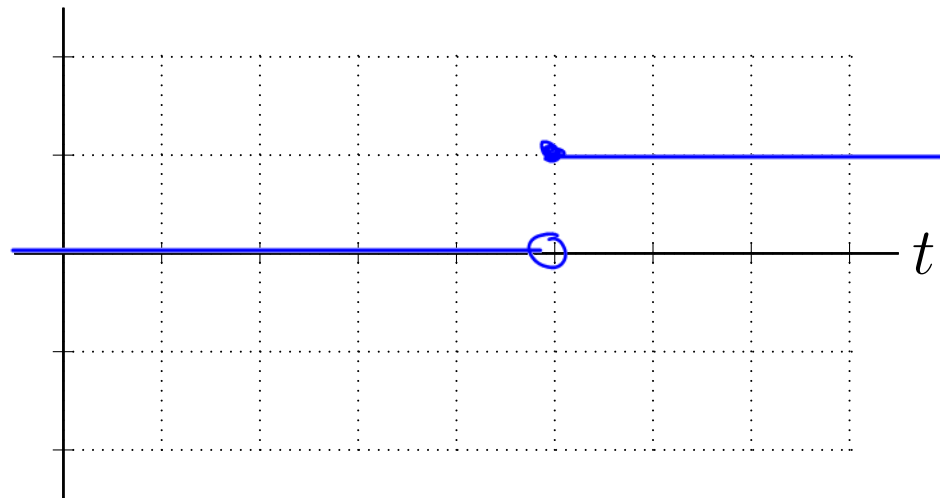
$$\text{by } u(t) := \begin{cases} 0 & \text{for } t < 0, & \text{off} \\ 1 & \text{for } t \geq 0. & \text{on} \end{cases}$$

Problem. Sketch the graph of $u(t)$.



$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

Problem. Sketch the graph of $u(t - 5)$.



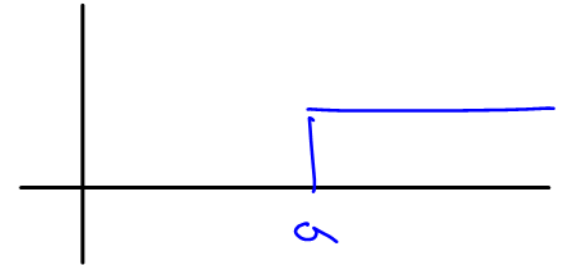
shift graph right by 5

Laplace Transform Using Step Functions

Problem. For $a > 0$, compute the Laplace transform of

$$u(t - a) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$$

$$\begin{aligned} \mathcal{L}(u(t-a)) &= \int_0^{\infty} e^{-st} u(t-a) dt \\ &\stackrel{\text{def}}{=} \int_a^{\infty} e^{-st} \cdot 1 dt \quad \left. \begin{array}{l} \text{for } t < a \\ u=0 \end{array} \right\} \\ &= \left. \begin{array}{l} \frac{e^{-st}}{-s} \Big|_a^{\infty} \\ \infty = t \\ a = t \end{array} \right\} \mathcal{L}\{1\} \\ &= \left(0 - \frac{e^{-s \cdot a}}{-s} \right) = \frac{1}{s} e^{-as} \quad \leftarrow \text{step function} \end{aligned}$$



Laplace Transform of Step Functions

$$\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$$

An alternate (and more directly useful form) is

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$



Notation: $u(t-a)$
 $= u_a(t)$

$$\mathcal{L}(u_a(t)f(t)) = e^{-as} \mathcal{L}(f(t+a))$$

t's replaced by t+a
no t's to replace

Problem. Find $\mathcal{L}(u_2)$.

$$\begin{aligned} \mathcal{L}\{u_2 \cdot 1\} &= e^{-2s} \mathcal{L}\{1\} \\ &= e^{-2s} \cdot \frac{1}{s} \end{aligned}$$

Problem. Find $\mathcal{L}(u_\pi)$.

$$\begin{aligned} \mathcal{L}\{u_\pi\} &= e^{-\pi s} \cdot \mathcal{L}\{1\} \\ &= e^{-\pi s} \cdot \frac{1}{s} \end{aligned}$$

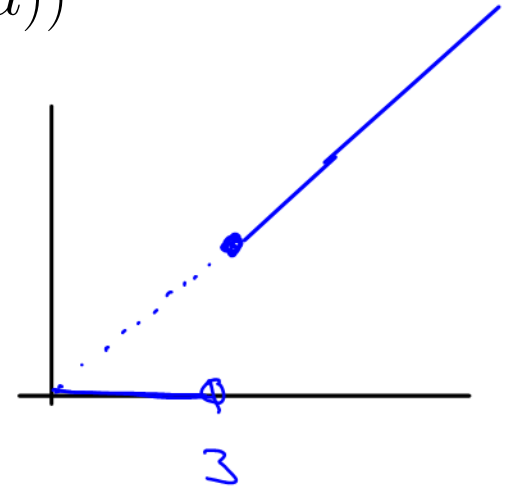
$$\mathcal{L}(u_a(t)f(t)) = e^{-as} \mathcal{L}(f(t+a))$$

Problem. Find $\mathcal{L}(tu_3)$.

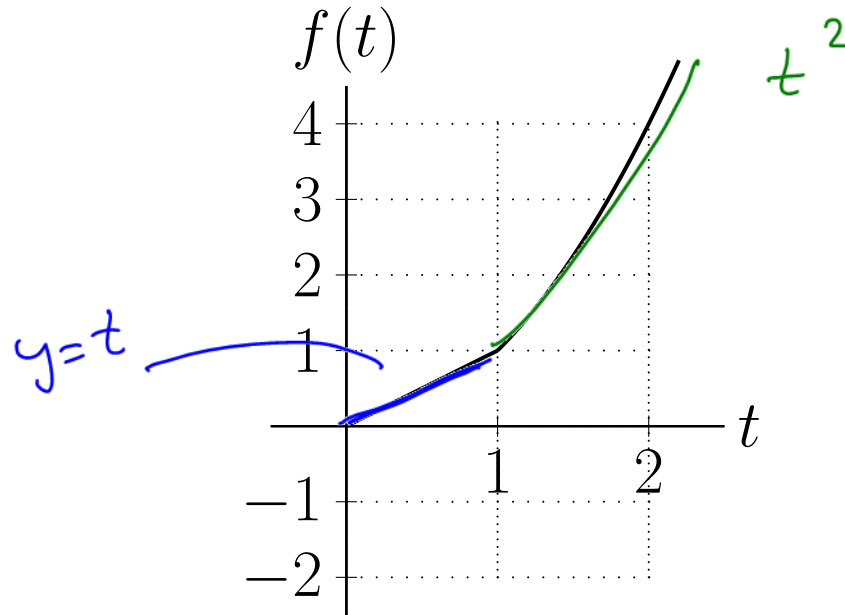
$$\begin{aligned} \mathcal{L}(u_3 t) &= e^{-3s} \mathcal{L}\{t+3\} \\ &= e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right) \end{aligned}$$

replace t 's by $t+3$

$$\begin{aligned} \mathcal{L}\{3\} &= \frac{3}{s} \\ \mathcal{L}\{t\} &= \frac{1!}{s^{1+1}} \end{aligned}$$



Problem. Here is a more complicated function made up of $f = t$ and $f = t^2$.



Write the function in piecewise form, and again using step functions.

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ t^2 & 1 < t \end{cases} = t(u_0 - u_1) + t^2 \cdot u_1$$

t "on" at $t=0$
 t "on" at $t=1$
 $t > 1, t \cdot (1-1) = t \cdot 0$
 write w/ u 's

$y = ?$

Problem. Find $\mathcal{L}(t(u_0 - u_1) + t^2 u_1)$.

$$= \mathcal{L}(t \cdot u_0) - \mathcal{L}(t \cdot u_1) + \mathcal{L}(t^2 \cdot u_1)$$

$$= \cancel{e^{-0s}} \cdot \mathcal{L}(t+0) - e^{-1s} \mathcal{L}(t+1)$$

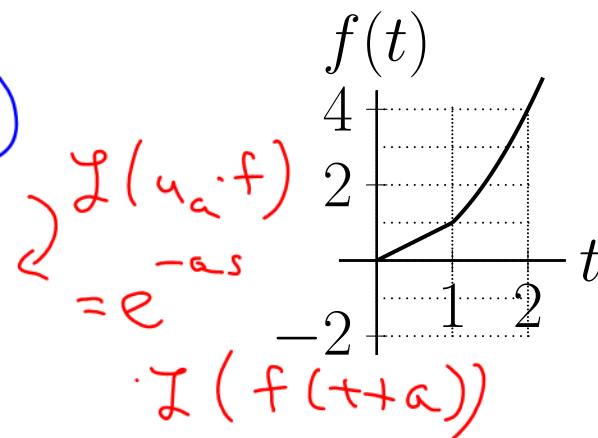
$$+ e^{-1s} \mathcal{L}((t+1)^2)$$

↓ expand

$$= \frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) + e^{-s} \mathcal{L}(t^2 + 2t + 1)$$

$$= \frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \cancel{\frac{1}{s}} \right) + e^{-s} \left(\frac{2}{s^3} + \cancel{2} \frac{1}{s^2} + \cancel{\frac{1}{s}} \right)$$

$$= \frac{1}{s} + e^{-s} \left(\frac{2}{s^3} + \frac{1}{s^2} \right)$$



Inverse Laplace Transform of Step Functions

$$\mathcal{L}^{-1} \left\{ \underbrace{e^{-as}}_{\substack{\text{a fact in table} \\ \swarrow}} F(s) \right\} = f(t-a)u_a$$

Problem. Find $\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2} \right\}$

$$= \mathcal{L}^{-1} \left\{ e^{-2s} \cdot \frac{1}{s^2} \right\}$$

\uparrow
 $\mathcal{L}(t)$

$\leftarrow F(s) = \frac{1}{s^2}$

$$= u_2(t) [t-2]$$

\uparrow
 step "on" at
 $t=2$

\swarrow
 t 's replaced by $t-a$

$$\mathcal{L}^{-1} \{ e^{-as} F(s) \} = f(t - a) u_a$$

Problem. Find $\mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s-4} \right\}$

$$= \mathcal{J}^{-1} \left\{ e^{-3s} \frac{1}{s-4} \right\}$$

$$\begin{array}{c} \uparrow \\ \mathcal{J}(e^{4t}) \end{array}$$

$$= u_{\underset{3}{3}}(t) \cdot e^{\underset{3}{4}(t-\underset{3}{3})}$$

$$\mathcal{L}^{-1} \{e^{-as} F(s)\} = f(t - a)u_a$$

Problem. Which of the following equals $f(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2 + 4} \right\}$?

1. $\frac{1}{s} \cos(\pi t)u_\pi$

2. $\frac{1}{\pi s} \cos(\pi(t - \pi))u_\pi$

3. $\frac{1}{2} \sin(2(t - \pi))u_\pi$

4. $\frac{1}{\pi} \sin(2(t - \pi))u_\pi$

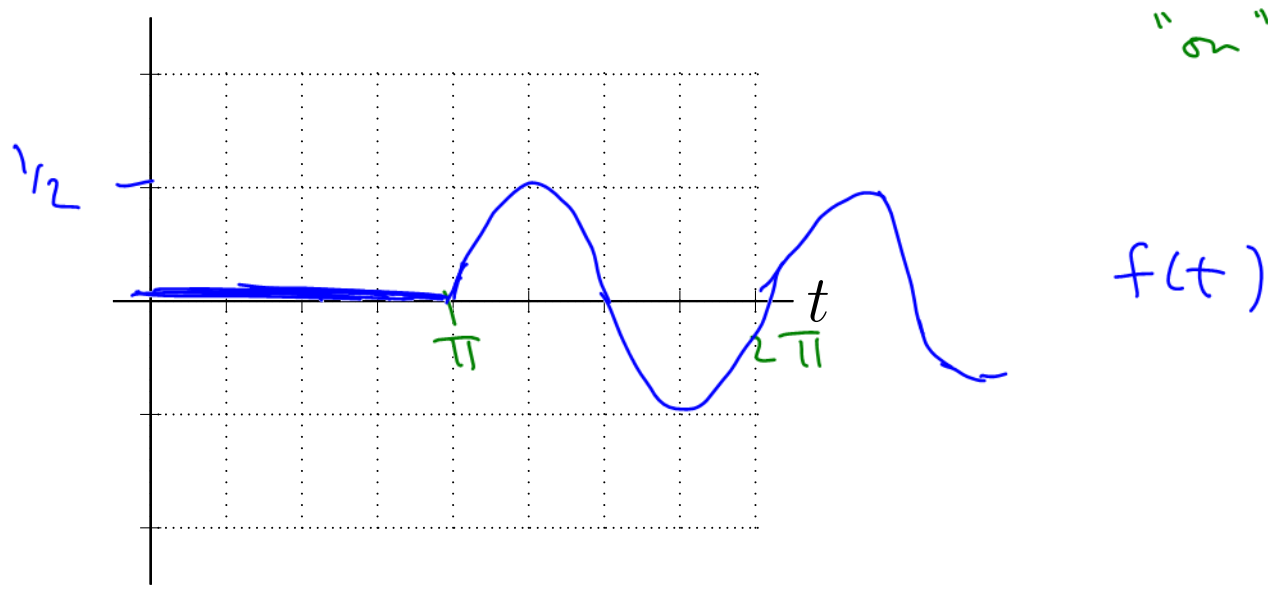
$$\begin{aligned} & \mathcal{L}^{-1} \left\{ e^{-\pi s} \cdot \frac{1}{2} \frac{2}{s^2 + 4} \right\} \\ &= \frac{1}{2} u_\pi(t) \sin(2(t - \pi)) \end{aligned}$$

$$\mathcal{L}^{-1} \left\{ \frac{k}{s^2 + k^2} \right\} = \sin(kt)$$

Problem. Sketch the graph of

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2 + 4} \right\} = \frac{1}{2} \sin(2(t - \pi)) u_\pi$$

freq \rightarrow period = $\frac{2\pi}{2} = \pi$
 "on" at $t = \pi$



Problem. Find $\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{(s-1)(s-2)} \right\}$

$$\mathcal{L}^{-1}\{e^{-as} \cdot F(s)\} = u_a \cdot f(t-a)$$

$$= \mathcal{L}^{-1} \left\{ e^{-2s} \cdot \left[\frac{1}{(s-1)(s-2)} \right] \right\}$$

↓ need part frac

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-1)$$

$$s=2$$

$$1 = B(2-1)$$

$$\boxed{B=1}$$

$$s=1$$


$$1 = A(1-2)$$

$$\boxed{A=-1}$$

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ e^{-2s} \left[\left(\frac{-1}{s-1} \right) + \left(\frac{1}{s-2} \right) \right] \right\} \\ &= \mathcal{L}^{-1} \left\{ e^{-2s} \frac{1}{s-1} \right\} + \mathcal{L}^{-1} \left\{ e^{-2s} \frac{1}{s-2} \right\} = -u_2 e^{1(t-2)} + u_2 e^{2(t-2)} \end{aligned}$$

Tips for Inverse Laplace With Step/Piecewise Functions

- Separate/group all terms by their e^{-as} factor.
- Complete any partial fractions **leaving the e^{-as} out front** of the term.
 - The e^{-as} only affects final inverse step.
 - Partial fraction decomposition only works for polynomial numerators.


$$\frac{10(e^{-10s})}{(s-1)(s-2)} \neq \frac{A}{s-1} + \frac{B}{s-2}$$
$$\checkmark = \left(\frac{A}{s-1} + \frac{B}{s-2} \right) \cdot e^{-10s}$$

The reason Laplace transforms can be helpful in solving differential equations is because there is a (relatively simple) transform rule for derivatives of functions.

Proposition (Differentiation). *If f is continuous on $[0, \infty)$, $f'(t)$ is piecewise continuous on $[0, \infty)$, and both functions are of exponential order a , then for $s > a$, we have*

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

s multiplier \mathcal{L} of orig'l

initial condition

Problem. Confirm the transform table entry for $\mathcal{L}\{\cos(kt)\}$ with the help of the transform derivative rule and the transform of $\sin(kt)$.

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{d}{dt} \frac{1}{k} \sin(kt)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{k} (\sin(kt))'\right\} = \frac{1}{k} \left[s \mathcal{L}\{\sin(kt)\} - \sin(0) \right]$$

\mathcal{L} for deriv

$$= \frac{1}{k} s \left[\frac{k}{s^2 + k^2} - 0 \right]$$

$$= \frac{s}{s^2 + k^2}$$

We can generalize this rule to the transform of higher derivatives of a function.

Theorem (General Differentiation). *If $f(t), f'(t), \dots, f^{(n-1)}(t)$ are continuous on $[0, \infty)$, $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, and all of these functions are of exponential order a , then for $s > a$, we have*

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

Can be proven using integration by parts n times.

n^{th} deriv

$F(s)$

$s^n \cdot$ Transform of orig'l
 f

Series of initial cond's

s power dec \rightarrow f deriv incr

Most commonly in this course, we will need specifically the transform of the second derivative of a function.

Corollary (Second Differentiation). *If $f(t)$ and $f'(t)$ are continuous on $[0, \infty)$, $f''(t)$ is piecewise continuous on $[0, \infty)$, and all of these functions are of exponential order a , then for $s > a$, we have*

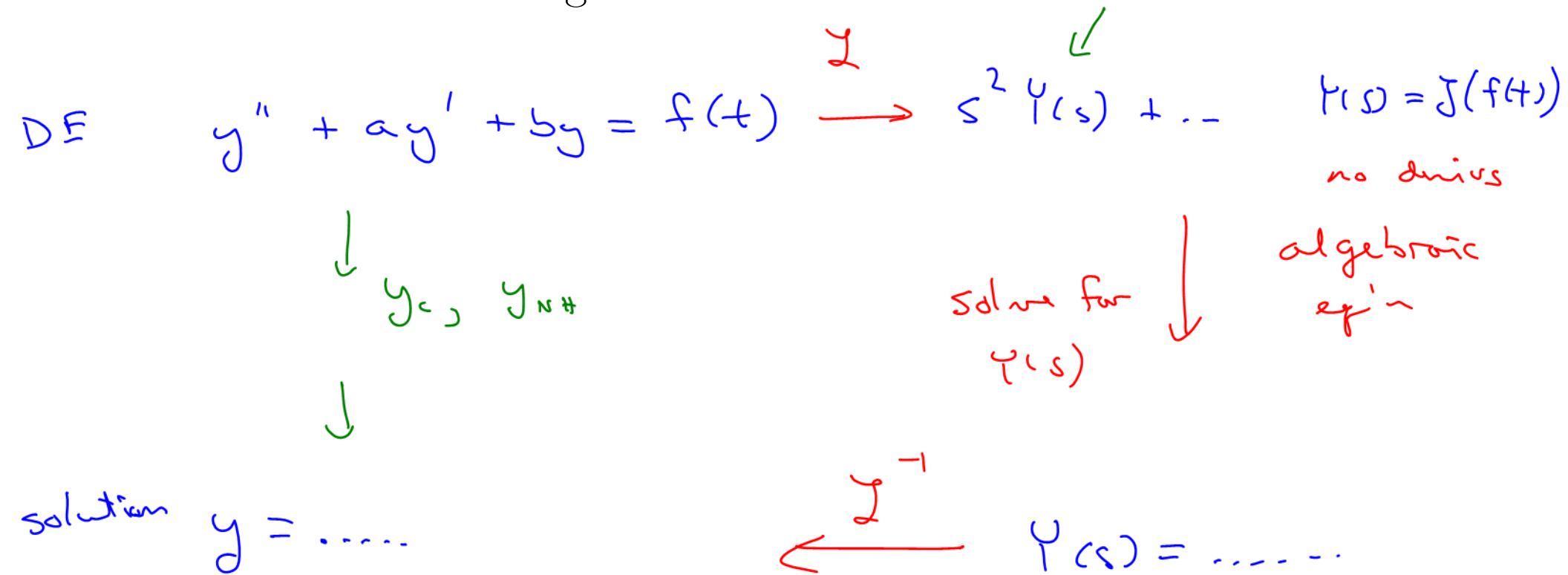
$$\mathcal{L}\{f''(t)\}(s) = s^2 \mathcal{L}\{f\}(s) - sf(0) - f'(0).$$

→
f deriv incr

Solving Initial Value Problems with Laplace Transforms

Problem. Sketch the general method.

$\mathcal{L}(y)$, unknown



Problem. Find the Laplace transform of the entire DE

$$\mathcal{L}(x' + x) = \mathcal{L}(\cos(2t)), x(0) = 0 \quad \mathcal{L}(x(t)) = X(s)$$

$$\underbrace{[sX(s) - x(0)]}_{\mathcal{L}(x')} + X(s) = \frac{s}{s^2 + 2^2}$$

$$sX(s) + X(s) = \frac{s}{s^2 + 4}$$

Problem. Note the form of the equation now: are there any derivatives left?

No!

Problem. Solve for $X(s)$.

$$sX(s) + X(s) = \frac{s}{s^2+4}$$

$$(s+1)X(s) = \frac{s}{s^2+4}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

Problem. Put $X(s)$ in a form so that you can find its inverse transform. \rightarrow partial fractions

$$\frac{s}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

$$s = (As+B)(s+1) + C(s^2+4)$$

$$s = -1$$

$$-1 = C(1+4)$$

$$\boxed{C = -1/5}$$

equate s^2

$$0 = A + C$$

$$A = -C$$

$$\boxed{A = 1/5}$$

const coeff's

$$0 = B + 4C$$

$$B = -4C$$

$$\boxed{B = 4/5}$$

$$X(s) = \frac{1}{5} \frac{s}{s^2+4} + \frac{4}{5} \frac{1}{s^2+4} + \frac{-1}{5} \frac{1}{s+1}$$

Problem. Find $x(t)$ by taking the inverse transform.

$$X(s) = \frac{1}{5} \frac{s}{s^2+4} + \frac{4}{5} \frac{1}{2} \frac{1 \times 2}{s^2+4} - \frac{1}{5} \frac{1}{s+1}$$

✓
 $\frac{2}{s^2+2^2}$
✓

✓
 $\frac{1}{s-a}$

\int ↓
 \int ↓
 \int ↓

$$x(t) = \frac{1}{5} \cos(2t) + \frac{2}{5} \sin(2t) - \frac{1}{5} e^{-t}$$

particular sol'n
 b/c we used initial condition $x(0) = 0$

Problem. Solve $y'' + y = \sin(2t)$, $y(0) = 2$, and $y'(0) = 1$.

\mathcal{L} of whole DE:

$$\mathcal{L}(y(t)) = Y(s)$$

$$\left[\begin{array}{l} s^2 Y(s) - s(2) - 1 \\ s^2 \mathcal{L}(y) - s y'(0) - y'(0) \end{array} \right] + Y(s) = \frac{2}{s^2+4}$$

Solve for $Y(s)$

$$(s^2+1) Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1) Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}$$

$$Y(s) = \frac{-\frac{2}{3} \downarrow B}{3} \frac{1}{2} \frac{x^2}{s^2+4} + \frac{2 \downarrow D}{3} \frac{1}{s^2+1} + \frac{2s+1}{s^2+1}$$

$$= -\frac{1}{3} \frac{2}{s^2+4} + 2 \frac{s}{s^2+1} + \frac{5}{3} \frac{1}{s^2+1}$$

$\mathcal{L}^{-1} \downarrow$

$$y(t) = -\frac{1}{3} \sin(2t) + 2 \cos(t) + \frac{5}{3} \sin(t)$$

particular solution.

$$\underline{y''} + y = \sin(2t), \quad y(0) = 2, \quad \text{and} \quad y'(0) = 1.$$

Problem. Confirm your solution is correct.

$$y = -\frac{1}{3} \sin(2t) + \underbrace{2 \cos(t) + \frac{5}{3} \sin(t)}_{y_c}$$

Need y' & y''

$$y' = -\frac{2}{3} \cos(2t) - 2 \sin(t) + \frac{5}{3} \cos(t) \quad \left. \begin{array}{l} \\ \end{array} \right\} \frac{d}{dt}$$

$$y'' = +\frac{4}{3} \sin(2t) - 2 \cos(t) - \frac{5}{3} \sin(t)$$

$$\text{LHS} = \underbrace{\left[\frac{4}{3} \sin(2t) - 2 \cos(t) - \frac{5}{3} \sin(t) \right]}_{y''} + \underbrace{\left[-\frac{1}{3} \sin(2t) + 2 \cos(t) + \frac{5}{3} \sin(t) \right]}_y$$

$$= 1 \sin(2t) = \text{RHS} \quad \checkmark$$

Problem. Solve $y'' - 2y' + 5y = -8e^{-t}$, $y(0) = 2$, and $y'(0) = 12$.

2^{nd} order

non-homog's

[could use
 $y_p \neq y_c$]

\mathcal{L} of DE

$$\underbrace{[s^2 Y(s) - s y(0) - y'(0)]}_{\mathcal{L}(y'')} - 2 \underbrace{[s Y(s) - y(0)]}_{\mathcal{L}(y')} + 5 Y(s) = -8 \frac{1}{s+1}$$

Gather $Y(s)$ terms

$$Y(s) [s^2 - 2s + 5] - s \cdot 2 - 12 - 2(-2) = -8 \frac{1}{s+1}$$

$$Y(s) [s^2 - 2s + 5] = \frac{-8}{s+1} + 2s + 8$$

$$Y(s) = \frac{-8}{(s+1)(s^2 - 2s + 5)} + \frac{2s + 8}{(s^2 - 2s + 5)}$$

$$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad \text{and} \quad y'(0) = 12.$$

$$Y(s) = \frac{-8}{(s+1)(s^2-2s+5)} + \frac{2s+8}{(s^2-2s+5)}$$

$$\frac{-8}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}$$

$$-8 = A(s^2-2s+5) + (Bs+C)(s+1)$$

$$s=-1 \quad -8 = A(1+2+5) \quad \boxed{A=-1}$$

$$s^2 \text{ coeff} \quad 0 = A+B \quad \boxed{B=1}$$

$$\text{const} \quad -8 = 5A+C \quad \boxed{C=-8-5A=-3}$$

$$Y(s) = \frac{-1}{s+1} + \frac{s}{s^2-2s+5} - 3 \frac{1}{s^2-2s+5} + \frac{2s+8}{s^2-2s+5}$$

$$= \frac{-1}{s+1} + 3 \frac{s}{s^2-2s+5} + 5 \frac{1}{s^2-2s+5}$$

$$Y(s) = \frac{-1}{s+1} + 3 \frac{s}{s^2-2s+5} + 5 \frac{1}{s^2-2s+5}$$

Good find $\mathcal{J}^{-1}(Y(s))$

$$Y(s) = \frac{-1}{s+1} + 3 \frac{s}{\underbrace{(s^2-2s+1)}_{(s-1)^2} - 1 + 5} + 5 \frac{1}{(s^2-2s+1) - 1 + 5}$$

$$Y(s) = \frac{-1}{s+1} + 3 \frac{(s-1)+1}{(s-1)^2+4} + 5 \frac{1}{(s-1)^2+4}$$

(2)

$$Y(s) = \frac{-1}{s+1} + 3 \frac{(s-1)}{(s-1)^2+2^2} + \frac{8}{2} \frac{1 \times 2}{(s-1)^2+2^2}$$

$$\textcircled{2} \mathcal{J}^{-1}\left(\frac{s}{s^2+2^2}\right)$$

$$= \cos(2t)$$

$$\mathcal{J}^{-1}(F(s-a))$$

$$= e^{at} \mathcal{J}^{-1}(F(s))$$

\downarrow
 \mathcal{J}^{-1}

\downarrow

$$y(t) = -e^{-t} + 3 e^t \cos(2t) + 4 e^t \sin(2t)$$

$$y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad \text{and} \quad y'(0) = 12.$$

Problem. Confirm your solution is correct.

$$y(t) = -e^{-t} + 3e^t \cos(2t) + 4e^t \sin(2t) \quad \downarrow \frac{d}{dt}$$

Need y', y'' for LHS:

$$\begin{aligned} y' &= e^{-t} + 3e^t \cos(2t) - 6e^t \sin(2t) + 4e^t \sin(2t) + 8e^t \cos(2t) \\ &= e^{-t} + 11e^t \cos(2t) - 2e^t \sin(2t) \quad \downarrow \frac{d}{dt} \end{aligned}$$

$$y'' = -e^{-t} + 11e^t \cos(2t) - 22e^t \sin(2t) - 2e^t \sin(2t) - 4e^t \cos(2t)$$

$\underbrace{\hspace{10em}}_{-24} = 7$

$$\begin{aligned} \text{LHS: } & (-e^{-t} + 7e^t \cos(2t) - 24e^t \sin(2t)) \\ & - 2(e^{-t} + 11e^t \cos(2t) - 2e^t \sin(2t)) \\ & + 5(-e^{-t} + 3e^t \cos(2t) + 4e^t \sin(2t)) \\ & = -8e^{-t} = \text{RHS} \quad \checkmark \end{aligned}$$

$$e^{-t} \text{ terms: } -1 - 2 - 5 = -8e^{-t}$$

$$e^t \cos(2t): 7 - 22 + 15 = 0$$

$$e^t \sin(2t): -24 + 4 + 20 = 0$$