Week #1 : Integration, Separation of Variables

Goals:

- Introduce differential equations.
- Review integration techniques.
- Solve first-order DEs using separation of variables.

Differential Equations

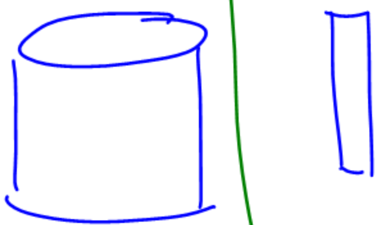
An equation relating an unknown function and one or more of its derivatives is a ***differential equation***.

We study differential equations because they are an incredibly powerful tool for modelling real world problems.

Problem. Express the following sentence as an equation:

The rate at which water is leaking from the tank is proportional to the square root of the volume of water in the tank.

"equals a constant times..."



Toricelli's law

derivative = $\boxed{\frac{dV}{dt} = k\sqrt{V}}$ equation w/ deriv.

define V = the volume of water in tank.

Goal: find V as a function of time

Problem. Express the following sentence as an equation:

The rate at which a rumour spreads is proportional to the product of the people who have heard it and those who have not.

deriv

let P = # people who have heard rumour. } varying w/time

Q = " " " have not " "

N = total # people → constant

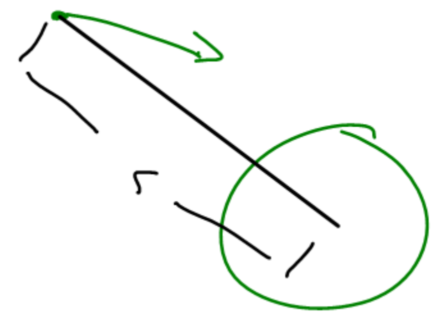
$$\frac{dP}{dt} = k\,P \cdot Q$$

$$\boxed{\frac{dP}{dt} = k\,P(N-P)}$$

Goal: P as function of time.

Problem. Express the following sentence as an equation:

As the meteorite plummets toward the Earth, its acceleration is (inversely proportional) to the (square of its distance) from the centre of the Earth.



$$\underset{\frac{d^2r}{dt^2}}{\cancel{a}} = k\,\frac{1}{r^2}$$

$$\boxed{\frac{d^2r}{dt^2} = k\,\frac{1}{r^2}}$$

The previous examples indicate how easily differential equations can be constructed. Unfortunately, starting with those equations, we have a lot of work to do before we can predict will happen, given the equation.

We now consider the most fundamental differential equation.

Problem. Consider

$$(*) \qquad \boxed{\frac{dy}{dt} = ky} \qquad \text{where } k \in \mathbb{R}.$$

What functions (if any) satisfy $(*)$ when $k = 1$?

Solutions to DEs

$$\frac{dy}{dt} = y \longrightarrow y = \dots \text{ as function of time?}$$

What $y =$ its own derivative?

$y = e^t$ then

$\frac{d}{dt}$

$$\frac{dy}{dt} = e^t \quad \text{so} \quad \frac{dy}{dt} = y \quad \text{for } \boxed{y = e^t}$$

solution.

$$(*)\ \frac{dy}{dt} = ky \qquad \text{where } k \in \mathbb{R}$$

For what values of k does a function y exists that satisfies $(*)$?

eg $\frac{dy}{dt} = 5y$ what y has

$\frac{dy}{dt} = 5$ (original y)

$y = e^{5t} \xrightarrow{d/dt} \frac{dy}{dt} = e^{5t} \cdot 5 = 5e^{5t} = 5y$

In general, $y = e^{kt}$ is a solution

$\frac{dy}{dt} = ky$

$$(*)\ \frac{dy}{dt} = ky \qquad \text{where } k \in \mathbb{R}$$

How many functions satisfy $(*)$ for each value of k? → an infinite number

$y = e^{kt}$ $\xrightarrow{\text{generalize}}$ $\boxed{y = Ce^{kt}}$ $C \in \mathbb{R}$

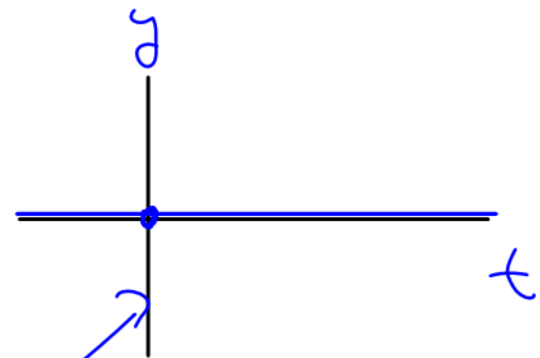
eg. $y = 10e^{kt}$ $\xrightarrow{d/dt}$ $\frac{dy}{dt} = 10e^{kt} \cdot k = k\underbrace{(10e^{kt})}_{y}$

$\frac{dy}{dt} = ky$ ✓

$$(*)\ \frac{dy}{dt} = ky \qquad \text{where } k \in \mathbb{R}$$

How many functions y are there such that y and $\frac{dy}{dt}$ are continuous for all real t, $\boxed{y(0) = 0,}$ and satisfy $(*)$? What are they?

initial conditions



$y = Ce^{kt}$

$y=0,\ t=0$

$0 = C\cancel{e^{0}}$ 1

$0 = C$

$\Rightarrow \boxed{y = 0}$

$y = 0 \Rightarrow \frac{dy}{dt} = 0$

(slope is always horiz'l)

$$(*)\ \frac{dy}{dt} = \underbrace{k}_{\text{given}}y \qquad \text{where } k \in \mathbb{R}$$

How many functions y are there such that y and $\dfrac{dy}{dt}$ are continuous for all real t, $\mathbf{y(1) = 1}$, and satisfy $(*)$? What are they?

$t=1,\ y=1$

$$y = Ce^{kt}$$

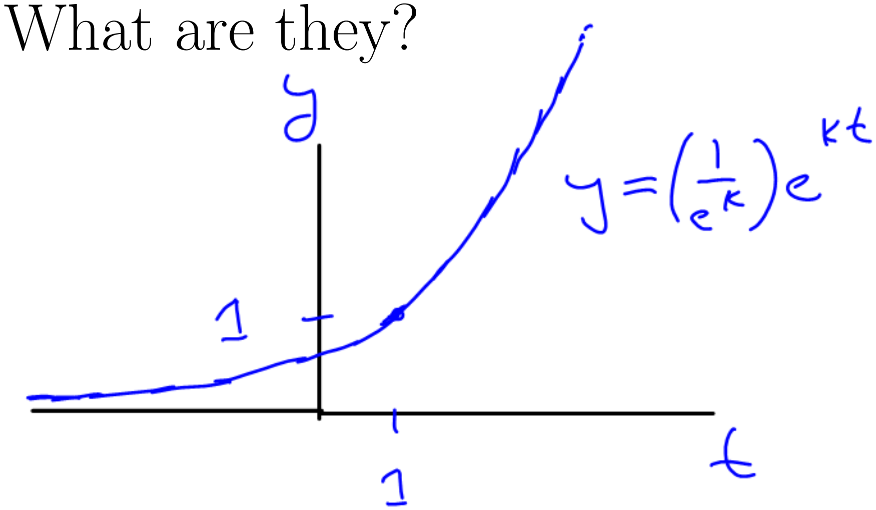
$$1 = Ce^{k\cdot 1}$$

$$1 = Ce^{k}$$

$$C = e^{-k} = \frac{1}{e^{k}}$$

$$y = \underbrace{\left(\frac{1}{e^{k}}\right)}_{\text{const}} \underbrace{e^{kt}}_{\text{exp'l}}$$

(k pos for graph)



The ***general solution*** is the family of all possible functions that satisfy a differential equation. It normally involves some parameters, i.e. arbitrary constants. ($y = \ldots$

A ***particular solution*** is a specific function with satisfies the differential equation.

Problem. Find the general solution of $\dfrac{dy}{dx} = \dfrac{1}{x^2 - 1}$.

$$\int \frac{dy}{dx}\,dx = y = \int \frac{1}{x^2-1}\,dx$$

dist. linear factors

$$y = \int \frac{1}{(x-1)(x+1)}\,dx \qquad \frac{A\,(x+1)}{(x-1)(x+1)} + \frac{B\,(x-1)}{(x+1)(x-1)} = \frac{1}{(x-1)(x+1)}$$

$$A(x+1) + B(x-1) = 1$$

sub $x = 1$: $A(1+1) + B(\cancel{1-1}) = 1$, $1-1 = 0$

$$2A = 1 \qquad \boxed{A = \tfrac{1}{2}}$$

$x = -1$: $B(-1-1) = 1 \qquad \boxed{B = -\tfrac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{x^2 - 1}$$

$$y = \int \frac{1}{(x-1)(x+1)}\, dx$$

partial fraction decomposition

$$= \int \left(\frac{1/2}{(x-1)} + \frac{(-1/2)}{(x+1)} \right) dx$$

integrate

$$y = \underbrace{\frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C}_{\text{family of solutions}}$$

satisfies

general solution.

Problem. Find the particular solution of $\dfrac{dy}{dx} = \dfrac{1}{x^2-1}$ which also satisfies $y(2) = 1$.

$x=2,\ y=1$

narrow down to a single/particular solution

$\Rightarrow$ find the value of C in general solution

$$y = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

solve for C

$$1 = \frac{1}{2}\cancel{\ln|2-1|} - \frac{1}{2}\ln|2+1| + C$$

$\ln(1) = 0$

$$1 + \frac{1}{2}\ln(3) = C$$

so $$\boxed{y = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \left(1 + \frac{1}{2}\ln(3)\right)}$$

particular sol'n satisfying $y(2)=1$ and DE.

A first-order ***initial value problem*** is a differential equation

$$y'(x) = F\big(x, y(x)\big)$$

together with a point (x_0, y_0) in the domain of F called the ***initial condition***. + DE = IVP

A ***solution*** to an initial value problem is a function y that satisfies the differential equation and $y(x_0) = y_0$. This solution with be a ***particular solution*** to differential equation.

Problem. Solve the initial value problem: $\underbrace{\frac{dy}{dt} = 2t + 1}_{\text{DE}}$; $\underbrace{y(0) = 3}_{\text{initial cond'n}}$.

DE + initial cond'n = IVP

Integrate both sides w.r.t. t.

$$y = \int (2t+1)\,dt$$

$$y = \frac{\cancel{2}t^2}{\cancel{2}} + t + C$$

general solution

$t=0$, $y=3$, Solve for C

$$3 = 0^2 + 0 + C \qquad C = 3$$

so $y = t^2 + t + 3$

is the particular solution to the I.V.P.

$$\frac{dy}{dt} = 2t + 1;\ y(0) = 3.$$

Problem. Solve the initial value problem: $\dfrac{dy}{dx} = x^2 e^x$; $y(0) = 2$.

Integrate both sides w.r.t. x

$y = \int x^2 e^x \, dx$

by parts

$u = x^2 \qquad \int dv = \int e^x \, dx$

$\downarrow d/dx$

$du = 2x \, dx \qquad v = e^x$

$y = x^2 e^x - \int e^x \, 2x \, dx$

$y = x^2 e^x - 2 \int x e^x \, dx$

by parts

$u = x \qquad \int dv = \int e^x \, dx$

$\downarrow d/dx$

$du = 1 \, dx \qquad v = e^x$

$y = x^2 e^x - 2\left[x e^x - \int e^x \, dx \right]$

$y = x^2 e^x - 2x e^x + 2e^x + C$

general solution.

$$\frac{dy}{dx} = x^2e^x;\ y(0) = 2.$$

$$y = x^2e^x - 2xe^x + 2e^x + C$$

general solution.

$$2 = \underbrace{\cancel{0^2e^0}}_{0} - \underbrace{\cancel{2\cdot 0\cdot e^0}}_{0} + 2\underbrace{\cancel{e^0}}_{1} + C$$

$$\boxed{C = 0}$$

so $\boxed{y = x^2e^x - 2xe^x + 2e^x}$ is the particular solution to this initial value problem. (IVP).

As with 'regular' (algebraic, or **non**-differential) equations, solving a differential equation from scratch can be very difficult. It is much easier to verify whether a possible solution (given or guessed at) is correct.

Problem. Determine which of the functions

1 2

$$y_1 = ke^t \text{ or } y_2 = e^{kt}$$

is actually a solution to $\dfrac{dy}{dt} = ky$.

Sub proposed sol'n into eq'n, check that LHS = RHS

1 need LHS = $\dfrac{dy}{dt}$

LHS $= \dfrac{d}{dt}(ke^t) = ke^t$; not equal

RHS $= ky = k(ke^t) = k^2e^t$

$\Rightarrow y_1 = ke^t$ is not a sol'n.

2 LHS $= \dfrac{d}{dt}(e^{kt}) = ke^{kt}$

RHS $= ky = k(e^{kt})$

are equal

$\Rightarrow y_2 = e^{kt}$ is a solution.

Problem. Show that $g(x) = \sin(x) - \cos(x)$ is a solution to the initial value problem:

"y"

$$\frac{d^2y}{dx^2} + y = 0 \qquad y(0) = -1 \qquad y'(0) = 1\,.$$

(1) (2) (3)

(2) LHS $= y(0) = \sin(0) - \cos(0) = -1 = \text{RHS}$ ✓

(3) Need $y' = g' = \cos(x) + \sin(x)$

LHS $= y'(0) = \cos(0) + \sin(0) = 1 = \text{RHS}$ ✓

take d/dx

(1) Need $y'' = -\sin(x) + \cos(x)$

$$\text{LHS} = \underbrace{\left[-\cancel{\sin(x)} + \cancel{\cos(x)}\right]}_{d^2y/dx^2} + \left[\cancel{\sin(x)} - \cancel{\cos(x)}\right]$$

$= 0 = \text{RHS}$ ✓

We can conclude that $\sin(x) - \cos(x)$ is a sol'n to the I.V.P.

Separable Equations

We now introduce our first technique for solving first-order differential equations.

If the right side of the equation $\frac{dy}{dx} = f(x, y)$ is the product of a function in x and a function in y, then the equation is ***separable***. By separating the variables and integrating, we obtain an implicit solution to the differential equation.

$\int \frac{x-5}{y^2}\,dx$ what $y = ?$

Problem. Solve $\frac{dy}{dx} = \underbrace{\frac{x-5}{y^2}}_{\text{mix of } x, y\text{'s}}$.

$$\int y^2\,dy = \int (x-5)\,dx$$

separate & integrate

$$\cancel{C} \quad \frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y^3 = \frac{3x^2}{2} - 15x + 3C$$

$$y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}$$

family of solutions / general sol'n

Verify solution $y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}$ $\quad \dfrac{dy}{dx} = \dfrac{x-5}{y^2}.$

sub into LHS, RHS of DE

LHS

$$= \frac{d}{dx}\left(\left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}\right)$$

$$= \frac{1}{3}\left(\frac{3x^2}{2} - 15x + 3c\right)^{-2/3} \cdot (3x - 15) = \frac{x-5}{(\ldots)^{2/3}}$$

$$\text{RHS} = \frac{x-5}{y^2} = \frac{x-5}{\left(\frac{3x^2}{2} - 15x + 3c\right)^{2/3}}$$

are equal

$$\Rightarrow \quad y = \left(\frac{3x^2}{2} - 15x + 3c\right)^{1/3}$$ is the correct sol'n.

Problem. Solve the initial value problem $\frac{dx}{dt} = \frac{x-1}{\underbrace{t+3}_{\text{mix } x, t\text{'s}}}$; $x(-1) = 0$.

Separate, integrate

Goal:

$x = $.

$$\int \underbrace{\frac{1}{x-1}}_{\substack{x\text{'s} \\ \text{no } t\text{'s}}} dx = \int \underbrace{\frac{1}{t+3}}_{\substack{t\text{'s} \\ \text{no } x\text{'s}}} dt$$

$$e^{(\ln|x-1|)} = e^{[\ln|t+3| + C]}$$

exponentiate

$$|x-1| = e^{\ln|t+3|} \cdot e^{C}$$

$$|x-1| = |t+3|\, e^{C}$$

take away abs values

define $A = \pm e^{C}$

$$x - 1 = (t+3)A$$

$$\boxed{x = A(t+3) + 1}$$ general solution.

$$\frac{dx}{dt} = \frac{x-1}{t+3};\ x(-1) = 0$$

$x = A(t+3)+1$ $t=-1,\ x=0$

$0 = A(-1+3)+1$

$2A = 1$ $A = \frac{1}{2}$

so $\boxed{x(t) = \frac{1}{2}(t+3)+1}$ part'r sol'n

Verify: $\text{LHS} = \frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{2}(t+3)+1\right)$

$= \frac{1}{2}$

$\text{RHS} = \frac{x-1}{t+3} = \frac{\overbrace{\left(\frac{1}{2}(t+3)\cancel{+1}\right)}^{x}\cancel{-1}}{t+3} = \frac{1}{2}\,\frac{\cancel{t+3}}{\cancel{t+3}}$

LHS = RHS ✓

Problem. Solve $\dfrac{dz}{dr} = \dfrac{6r^5 - 2r + 1}{\cos(z) + e^z}$.

Goal: $z = \ldots$

$$\int (\cos(z) + e^z)\, dz = \int (6r^5 - 2r + 1)\, dr$$

Separate & integrate

$$\underbrace{\sin(z) + e^z}_{\text{can't isolate for } z = \ldots} = \frac{6r^6}{6} - \frac{2r^2}{2} + r + C$$

} implicit solution.

can't isolate

for $z = \ldots$

$$\frac{dz}{dr} = \frac{6r^5 - 2r + 1}{\cos(z) + e^z}.$$

$$\frac{d}{dr}\left(\sin(z) + e^z\right) = \frac{d}{dr}\left(r^6 - r^2 + r + C\right)$$

Verify: use implicit deriv's.

$$\text{LHS} = \frac{dz}{dr}$$

$$: \quad \cos(z)\frac{dz}{dr} + e^z \cdot \frac{dz}{dr} = 6r^5 - 2r + 1$$

$$\text{LHS} = \frac{dz}{dr} = \frac{6r^5 - 2r + 1}{(\cos(z) + e^z)} = \text{RHS} \checkmark$$

Problem. Solve $\dfrac{dy}{dx} = ky$. (know $y = Ce^{kt}$)

This equation describes exponential growth or decay.

Separate & integrate

$$\int \frac{1}{y}\, dy = \int k\, dx$$

$$e^{(\ln|y|)} = e^{(kx + C)}$$ exponentiate

$$|y| = e^{kx} \cdot e^{C}$$ remove abs value

$$A = \pm e^{C}$$

$$y = e^{kx} \cdot A$$

$$\boxed{y = Ae^{kx}}$$

Problem. Newton's law of cooling states that the rate of change in the temperature H of a body is proportional to the difference between its temperature H and that of its surroundings S. Solve

$$\frac{dH}{dt} = k(\underbrace{S}_{\text{const}} - H)$$

$H(t) =$ prediction of object's temp over time

Separate & Integrate

$$\int \frac{1}{S-H}\, dH = \int k \, dt$$

$$\ln|S-H|(-1) = kt + C$$

$$e^{(\ln|S-H|)} = e^{(-kt-C)}$$

$$|S-H| = e^{-kt} \cdot e^{-c}$$

let $A = \pm e^{-c}$

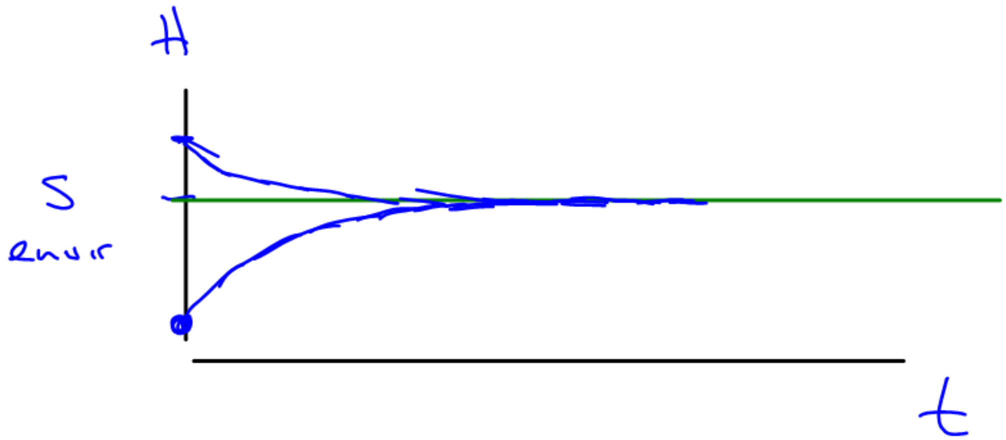
$$S-H = Ae^{-kt}$$

$$\boxed{H = S - Ae^{-kt}}$$

gen'l sol'n.

$$\frac{dH}{dt} = k(S - H)$$

What happens to the temperature H as $t \to \infty$?



$$H = S - \underbrace{Ae^{-kt}}$$

as $t \to \infty$

$$e^{-kt} \to 0$$

$\Rightarrow H \to S$, env temp

obj temp

Problem. Solve the explosion equation: $\frac{dx}{dt} = x^2$ where $x(0) > 0$. Give a physical interpretation of the solution.

This equations occurs in chemistry when the rate of a reaction is proportional to the concentration of the two reagents.

Separate & integrate

$$\int \frac{1}{x^2}\,dx = \int 1 \cdot dt$$

x^{-2}

$$\frac{x^{-1}}{-1} = t + C$$

$$\frac{1}{x} = -t - C$$

$$\boxed{\frac{1}{-t-C} = x} \checkmark$$

gener'l sol'n

Verify: LHS $= \frac{dx}{dt}$

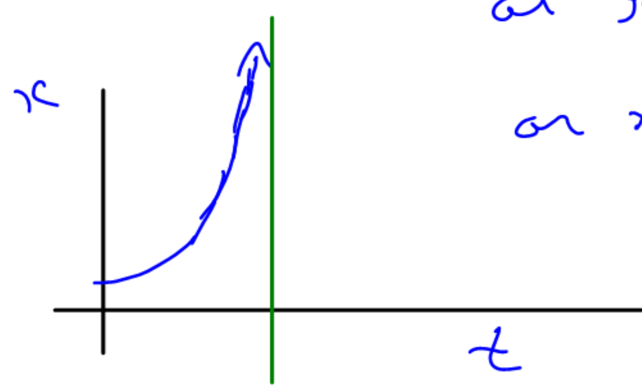
$= \frac{d}{dt}\left(\frac{1}{-t-c}\right) = \frac{d}{dt}(-t-c)^{-1}$

$= (-1)(-t-c)^{-2}(-1)$

$= \frac{1}{(-t-c)^2}$

RHS $= x^2 = \frac{1}{(-t-c)^2}$ ✓

LHS = RHS



at some t, $x \to \infty$

or $x(t)$ has a vert'l asymptote.

Problem. Solve the logistic equation: $\dfrac{dP}{dt} = (1-P)P$.

$P = \ldots$

This equation is used to model reproduction while taking competition into account.

rumour
$$\frac{dP}{dt} = k(P)(N-P)$$

Separate & Integrate

$$\int \frac{1}{(1-P)P}\,dP = \int dt$$

partial fractions $= t + C$

$$\frac{1}{(1-P)P} = \frac{A}{1-P} + \frac{B}{P}$$

$$\cancel{\frac{1}{(1-P)P}} = \frac{AP + B(1-P)}{\cancel{(1-P)P}} \quad \text{for all } P$$

$$\int \left(\frac{1}{1-P} + \frac{1}{P}\right) dP = t + C$$

$$\frac{\ln|1-P|}{-1} + \ln|P| = t + C$$

$P = 0 \to 1 = B(1) \qquad B = 1$

$P = 1 \to 1 = A(1) \qquad A = 1$

$$\ln|1-P| - \ln|P| = -t - C$$

$$\ln\left|\frac{1-P}{P}\right| = -t - C$$

$$\ln\left|\frac{1}{P} - 1\right| = -t - C$$

$$\frac{dP}{dt} = (1-P)P$$

$$e^{\ln\left|\frac{1}{P}-1\right|} = e^{(-t-C)}$$

$$\left|\frac{1}{P}-1\right| = e^{-t}\cdot e^{-C}$$

let $A = \pm e^{-C}$

$$\frac{1}{P}-1 = Ae^{-t}$$

$$\frac{1}{P} = Ae^{-t}+1$$

$$\boxed{\frac{1}{1+Ae^{-t}} = P}$$

gen'l solution

$$\frac{dP}{dt} = (1-P)P$$

$$P = \frac{1}{1+Ae^{-t}}$$

$$LHS = \frac{dP}{dt} = -1\left(1+Ae^{-t}\right)^{-2}\left(Ae^{-t}(-1)\right)$$

$$= \frac{Ae^{-t}}{(1+Ae^{-t})^2}$$

$$RHS = (1-P)(P) = \underbrace{\left(1-\frac{1}{1+Ae^{-t}}\right)}_{1-P}\underbrace{\left(\frac{1}{1+Ae^{-t}}\right)}_{P}$$

$$= \left(\frac{1+Ae^{-t}-1}{1+Ae^{-t}}\right)\left(\frac{1}{1+Ae^{-t}}\right) = \frac{Ae^{-t}}{(1+Ae^{-t})^2}$$

LHS = RHS ✓

confirmed sol'n is correct.