Week #11

Some problems and solutions selected or adapted from Stewart Calculus.

Triple Integrals

1. Evaluate the triple integral

   (a) $\iiint_E 6xy\, dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

   (b) $\iiint_E x\ dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

(a) Here

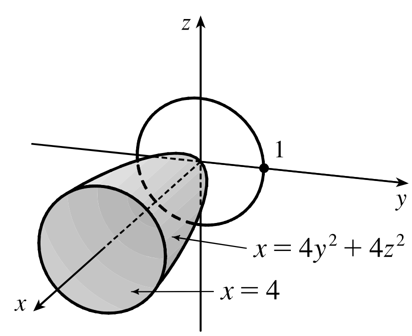
$$E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, \\ 0 \le z \le 1 + x + y\},$$

so

$$\begin{aligned}
&\iiint_E 6xy\, dV \\
&= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\, dz\, dy\, dx \\
&= \int_0^1 \int_0^{\sqrt{x}} [6xyz]_{z=0}^{z=1+x+y}\, dy\, dx \\
&= \int_0^1 \int_0^{\sqrt{x}} 6xy(1 + x + y)\, dy\, dx \\
&= \int_0^1 \left[4xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}}\, dx \\
&= \int_0^1 \left(3x^2 + 3x^3 + 2x^{5/2}\right)\, dx \\
&= \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 \\
&= \frac{65}{28}
\end{aligned}$$

(b) The projection of E on the yz-plane is the disk $y^2 + z^2 \le 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\begin{aligned}
&\iiint_E x\ dV \\
&= \iint_D \left[\int_{4y^2+4z^2}^4 x\, dx\right]\, dA \\
&= \frac{1}{2}\iint_D \left[4^2 - \left(4y^2 + 4z^2\right)^2\right]\, dA \\
&= 8\int_0^{2\pi}\int_0^1 \left(1 - r^4\right) r\, dr\, d\theta \\
&= 8\int_0^{2\pi} d\theta \int_0^1 \left(r - r^5\right)\, dr \\
&= 8(2\pi)\left[\frac{1}{2}r^2 - \frac{1}{6}r^6\right]_0^1 \\
&= \frac{16\pi}{3}
\end{aligned}$$



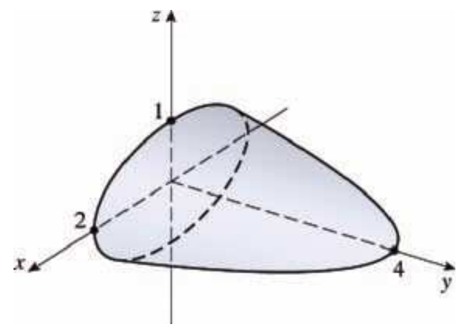
2. Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

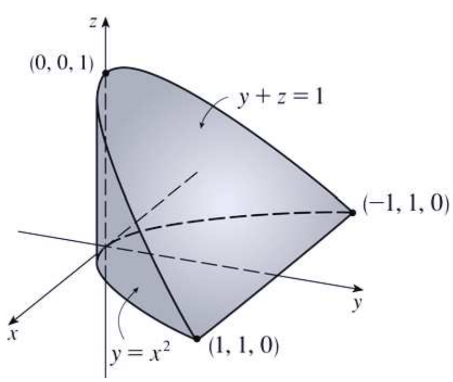
The plane $y + z = 1$ intersects the xy-plane in the line $y = 1$, so

$$E = \left\{(x, y, z) \mid -1 \le x \le 1, x^2 \le y \le 1, \\ 0 \le z \le 1 - y\right\}$$

and

$$\begin{aligned}
V &= \iiint_E dV \\
&= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx \\
&= \int_{-1}^{1}\int_{x^2}^{1}(1-y)\,dy\,dx \\
&= \int_{-1}^{1}\left[y-\frac{1}{2}y^2\right]_{y=x^2}^{y=1} dx \\
&= \int_{-1}^{1}\left(\frac{1}{2}-x^2+\frac{1}{2}x^4\right) dx \\
&= \left[\frac{1}{2}x-\frac{1}{3}x^3+\frac{1}{10}x^5\right]_{-1}^{1} \\
&= \frac{1}{2}-\frac{1}{3}+\frac{1}{10}+\frac{1}{2}-\frac{1}{3}+\frac{1}{10} \\
&= \frac{8}{15}
\end{aligned}$$

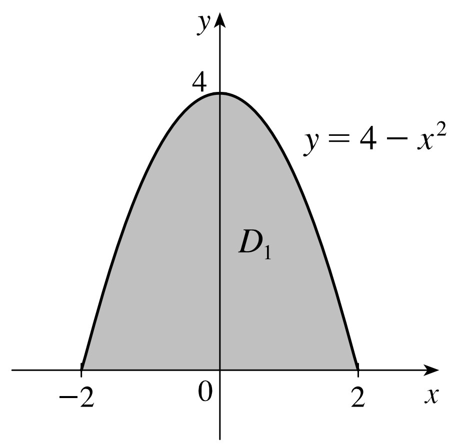


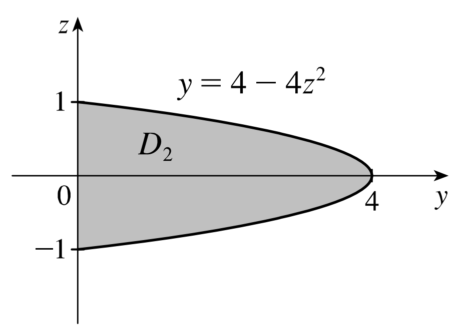


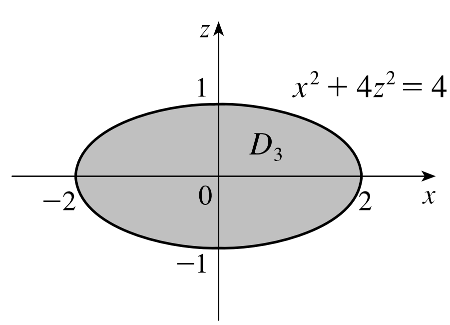
3. Express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by $y=4-x^2-4z^2$, $y=0$.

If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$\begin{aligned}
D_1 &= \{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4-x^2\} \\
&= \left\{(x,y) \mid 0 \le y \le 4, -\sqrt{4-y} \le x \le \sqrt{4-y}\right\} \\
D_2 &= \left\{(y,z) \mid 0 \le y \le 4, -\frac{1}{2}\sqrt{4-y} \le z \le \frac{1}{2}\sqrt{4-y}\right\} \\
&= \{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4-4z^2\} \\
D_3 &= \{(x,z) \mid x^2+4z^2 \le 4\}
\end{aligned}$$







Therefore

$$
\begin{aligned}
E = \Big\{(x,y,z) \mid & -2 \le x \le 2, \\
& 0 \le y \le 4 - x^2, \\
& -\frac{1}{2}\sqrt{4 - x^2 - y} \le z \le \frac{1}{2}\sqrt{4 - x^2 - y}\Big\} \\
= \{(x,y,z) \mid & \ 0 \le y \le 4, \\
& -\sqrt{4-y} \le x \le \sqrt{4-y}, \\
& -\frac{1}{2}\sqrt{4 - x^2 - y} \le x \le \frac{1}{2}\sqrt{4 - x^2 - y}\Big\} \\
= \{(x,y,z) & -1 \le z \le 1 \\
& 0 \le y \le 4 - 4z^2, \\
& -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\Big\} \\
= \{(x,y,z) \mid & \ 0 \le y \le 4, \\
& -\frac{1}{2}\sqrt{4-y} \le z \le \frac{1}{2}\sqrt{4-y}, \\
& -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\Big\} \\
= \{(x,y,z) \mid & -2 \le x \le 2, \\
& -\frac{1}{2}\sqrt{4-x^2} \le z \le \frac{1}{2}\sqrt{4-x^2}, \\
& 0 \le y \le 4 - x^2 - 4z^2\} \\
= \{(x,y,z) \mid & -1 \le z \le 1, \\
& -\sqrt{4-4z^2} \le x \le \sqrt{4-4z^2}, \\
& 0 \le y \le 4 - x^2 - 4z^2\}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\iiint_E f(x,y,z)\, dV \\
&= \int_{-1}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\, dz\, dy\, dx \\
&= \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\, dz\, dx\, dy \\
&= \int_{-1}^{1}\int_{0}^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\, dx\, dy\, dz \\
&= \int_{0}^{4}\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\, dx\, dz\, dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{0}^{4-x^2-4z^2} f(x,y,z)\, dy\, dz\, dx \\
&= \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{0}^{4-x^2-4z^2} f(x,y,z)\, dy\, dx\, dz
\end{aligned}
$$

Triple Integrals in Cylindrical Coordinates

4. Evaluate $\iiint_E (x+y+z)\,dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \Rightarrow r = 2$, so in cylindrical coordinates, E is given by

$$\left\{(r, \theta, z) \mid 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}.$$

Thus

$$\begin{aligned}
&\iiint_E (x + y + z)\,dV \\
&= \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)r\,dz\,dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \frac{1}{2}rz^2\right]_{z=0}^{z=4-r^2} dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^2 \left[\left(4r^2 - r^4\right)(\cos\theta + \sin\theta)\right. \\
&\quad \left. + \frac{1}{2}r\left(4 - r^2\right)^2\right] dr\,d\theta \\
&= \int_0^{\pi/2} \left[\left(\frac{4}{3}r^3 - \frac{1}{5}r^5\right)(\cos\theta + \sin\theta)\right. \\
&\quad \left. - \frac{1}{12}\left(r - r^2\right)^3\right]_{r=0}^{r=2} d\theta \\
&= \int_0^{\pi/2} \left[\frac{64}{15}(\cos\theta + \sin\theta) + \frac{16}{3}\right] d\theta \\
&= \left[\frac{64}{15}(\sin\theta - \cos\theta) + \frac{16}{3}\theta\right]_0^{\pi/2} \\
&= \frac{64}{15}(1 - 0) + \frac{16}{3}\cdot\frac{\pi}{2} - \frac{64}{15}(0 - 1) - 0 \\
&= \frac{83}{\pi} + \frac{128}{15}
\end{aligned}$$

5. Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies within the cyliner $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$$

and

$$\begin{aligned}
&\iiint_E x^2\,dV \\
&= \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2\cos^2\theta r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 \left[r^3\cos^2\theta z\right]_{z=0}^{z=0} dr\,d\theta \\
&= \int_0^{2\pi} \int_0^1 2r^4\cos^2\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1} d\theta \\
&= \frac{2}{5}\int_0^{2\pi} \cos^2\theta\,d\theta \\
&= \frac{2}{5}\int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta)\,d\theta \\
&= \frac{1}{5}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} \\
&= \frac{2\pi}{5}
\end{aligned}$$

6. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 2$.

In cylindrical coordinates, E is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The cone and the sphere intersect when $2r^2 = 2 \Rightarrow r = 1$, so

$$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{2-r^2}\right\}$$

and the volume is

$$\begin{aligned}
&\iiint_E dV \\
&= \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{\sqrt{2-r^2}}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2} - r^2\right)\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 \left(r\sqrt{2-r^2} - r^2\right)\,dr \\
&= 2\pi\left[-\frac{1}{3}\left(2-r^2\right)^{3/2} - \frac{1}{3}r^3\right]_0^1 \\
&= 2\pi\left(-\frac{1}{3}\right)\left(1+1-2^{3/2}\right) \\
&= -\frac{2}{3}\pi\left(2-\sqrt{2}\right) \\
&= \frac{4}{3}\pi\left(\sqrt{2}-1\right)
\end{aligned}$$

7. Find the mass and center of mass of the solid S bounded by the paraboloid $z = 4x^2+4y^2$ and the plane $z = a$ $(a > 0)$ if S has constant density K.

The paraboloid $z = 4x^2+4y^2$ intersects the plane $z = a$ when $a = 4x^2+4y^2$ or $x^2+y^2 = \frac{1}{4}a$. So, in cylindrical coordinates,

$$E = \left\{(r,\theta,z) \mid 0 \le r \le \frac{1}{2}\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\right\}.$$

Thus

$$\begin{aligned}
m &= \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta \\
&= K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(ar - 4r^3\right)\,dr\,d\theta \\
&= K\int_0^{2\pi}\left[\frac{1}{2}ar^2 - r^4\right]_{r=0}^{r=\sqrt{a}/2} d\theta \\
&= K\int_0^{2\pi}\frac{1}{16}a^2\,d\theta \\
&= \frac{1}{8}a^2\pi K
\end{aligned}$$

Since the region is homogeneous and symmetric, $\bar{x} = \bar{y} = 0$.

The z coordinate for the center of mass does require some calculation:

$$\begin{aligned}
\bar{z} &= \frac{1}{m}\int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} \\
&= \left(\frac{1}{m}\right)K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\frac{1}{2}a^2r - 8r^5\right)\,dr\,d\theta \\
&= \left(\frac{1}{m}\right)K\int_0^{2\pi}\left[\frac{1}{4}a^2r^2 - \frac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2} d\theta \\
&= \left(\frac{1}{m}\right)K\int_0^{2\pi}\frac{1}{24}a^3\,d\theta \\
&= \left(\frac{1}{m}\right)\frac{1}{12}a^3\pi K \\
&= \left(\frac{1}{\frac{1}{8}a^2\pi K}\right)\frac{1}{12}a^3\pi K \\
&= \frac{2}{3}a
\end{aligned}$$

Hence $(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{2}{3}a\right)$.