# Week #10

Some problems and solutions selected or adapted from Stewart Calculus.

## Double Integrals in Polar Coordinates

1. Evaluate the given integral by changing to polar coordinates.
   (a) $\iint\limits_D x^2 y \, dA$, where $D$ is the top half of the disk with center the origin and radius 5
   (b) $\iint\limits_R \sin\left(x^2+y^2\right) \; dA$, where $R$ is the region in the first quadrant between the circles with center the origin and radii 1 and 3
   (c) $\iint De^{-x^2-y^2} \; dA$, where $D$ is the region bounded by the semicircle $x = \sqrt{4-y^2}$ and the $y$-axis

(a) The half disk $D$ can be described in polar coordinates as

$$D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}.$$

Then

$$\begin{aligned}
\iint\limits_D &= \int_0^\pi \int_0^5 (r\cos\theta)^2 (r\sin\theta)\, r \; dr \; d\theta \\
&= \int_0^\pi \int_0^5 \cos^2\theta \sin\theta \; r^4 \; dr \; d\theta \\
&= \int_0^\pi \cos^2\theta \sin\theta \left[\frac{1}{5}r^5\right]_0^5 \; d\theta \\
&= \int_0^\pi \cos^2\theta \sin\theta \cdot 625 \; d\theta \\
\text{by substitution:} \quad &= \left(-\frac{1}{3}\right)\cos^3\theta \cdot 625\Big|_0^\pi \\
&= -\frac{1}{3}(-1-1)\cdot 625 \\
&= \frac{1250}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
&\iint\limits_R \sin\left(x^2+y^2\right) \; dA \\
&= \int_0^{\pi/2} \int_1^3 \sin\left(r^2\right) r \; dr \; d\theta \\
&= \int_0^{\pi/2} \left(\int_1^3 r\sin\left(r^2\right) \; dr\right) d\theta \\
\text{by substitution:} \quad &= \int_0^{\pi/2} \left[-\frac{1}{2}\cos\left(r^2\right)\right]_1^3 \; d\theta \\
&= \int_0^{\pi/2} \left(-\frac{1}{2}\right)\left[(\cos 9 - \cos 1)\right] \; d\theta \\
&= \frac{1}{2}(\cos 1 - \cos 9)\,\theta\Big|_0^{2\pi} = \frac{\pi}{4}(\cos 1 - \cos
\end{aligned}$$

(c)

$$\begin{aligned}
\iint\limits_D e^{-x^2-y^2} \; dA &= \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r \; dr \; d\theta \\
&= \int_{-\pi/2}^{\pi/2} \int_0^2 re^{-r^2} \; dr \; d\theta \\
\text{by substitution:} \quad &= \int_{-\pi/2}^{\pi/2} \left[-\frac{1}{2}e^{-r^2}\right]_0^2 \; d\theta \\
&= \int_{-\pi/2}^{\pi/2} \left(-\frac{1}{2}\right)\left(e^{-4}-e^0\right) \; d\theta \\
&= \frac{1}{2}\left(1-e^{-4}\right)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{\pi}{2}\left(1-e^{-4}\right)
\end{aligned}$$

2. Use a double integral to find the area of the region inside one loop of the rose $r = \cos 3\pi$.

One loop is given by the region

$$D = \{(r,\theta) \mid ; -\pi/6 \le \theta \le \pi/6, 0 \le r \le \cos 3\theta\},$$

so the area is

$$\begin{aligned}
\iint\limits_D \; dA &= \int_{-\pi/6}^{\pi/6} \int_0^{\cos 3\theta} r \; dr \; d\theta \\
&= \int_{-\pi/6}^{\pi/6} \left[\frac{1}{2}r^2\right]_{r=0}^{r=\cos 3\theta} \; d\theta \\
&= \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta \; d\theta
\end{aligned}$$

Since this function is even (symmetrical across $\theta = 0$), we can just integrate half the interval, and then multiply that by 2. We save a bit of work later, because 0 is a simpler limit of integration that $\frac{-\pi}{6}$.

$$I = \int_{-\pi/6}^{\pi/6} \frac{1}{2} \cos^2 3\theta \, d\theta$$

$$\begin{aligned}
I &= \mathbf{2} \int_{\mathbf{0}}^{\pi/6} \frac{1}{2} \cos^2 3\theta \, d\theta \\
&= 2 \int_0^{\pi/6} \frac{1}{2} \left( \frac{1 + \cos 6\theta}{2} \right) \, d\theta \\
&= \frac{1}{2} \left[ \theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} \\
&= \frac{\pi}{12}
\end{aligned}$$

3. Use polar coordinates to find the volume of the given solid.
   (a) Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leq 4$
   (b) Inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$

(a)

$$\begin{aligned}
V &= \iint_{x^2+y^2 \leq 4} \sqrt{x^2 + y^2} \, dA \\
&= \int_0^{2\pi} \int_0^2 \sqrt{r^2} r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^2 r^2 \, dr \\
&= [\theta]_0^{2\pi} \left[ \frac{1}{3} r^3 \right]_0^2 \\
&= 2\pi \left( \frac{8}{3} \right) \\
&= \frac{16}{3} \pi
\end{aligned}$$

(b) The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$\begin{aligned}
V &= \iint_{x^2+y^2 \leq 4} \left[ \sqrt{64 - 4x^2 - 4y^2} \right. \\
&\qquad \left. - \left( -\sqrt{64 - 4x^2 - 4y^2} \right) \right] \, dA \\
&= \iint_{x^2+y^2 \leq 4} 2\sqrt{64 - 4x^2 - 4y^2} \, dA \\
&= 4 \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} \, r \, dr \, d\theta \\
&= 4 \int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2} \, dr \\
&= 4[\theta]_0^{2\pi} \left[ -\frac{1}{3} \left( 16 - r^2 \right)^{3/2} \right]_0^2 \\
&= 8\pi \left( -\frac{1}{3} \right) \left( 12^{3/2} - 16^{2/3} \right) \\
&= \frac{8\pi}{3} \left( 64 - 24\sqrt{3} \right)
\end{aligned}$$

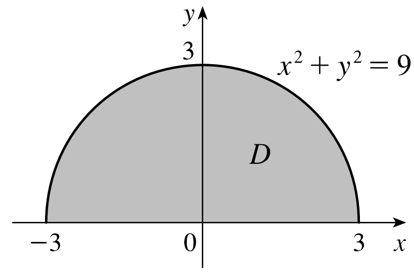
4. Evaluate the iterated integral by converting to polar coordinates.

   (a) $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin\left(x^2+y^2\right)\,dy\,dx$

   (b) $\displaystyle\int_{0}^{1}\int_{y}^{\sqrt{2-y^2}} (x+y)\,dx\,dy$

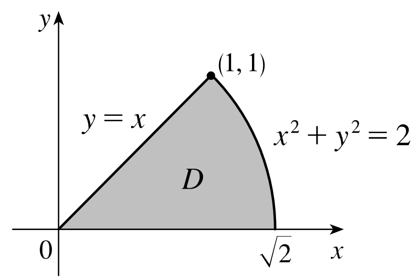
(a)

$$\begin{aligned}
&\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin\left(x^2+y^2\right)\,dy\,dx\\
&=\int_{0}^{\pi}\int_{0}^{3} \sin\left(r^2\right) r\,dr\,d\theta\\
&=\int_{0}^{\pi} d\theta \int_{0}^{3} r\sin\left(r^2\right)\,dr\\
&=[\theta]_0^{\pi}\left[-\frac{1}{2}\cos\left(r^2\right)\right]_0^3\\
&=\pi\left(-\frac{1}{2}\right)(\cos 9-1)\\
&=\frac{\pi}{2}(1-\cos 9)
\end{aligned}$$




(b)

$$\begin{aligned}
&\int_{0}^{\pi/4}\int_{0}^{\sqrt{2}} (r\cos\theta+r\sin\theta)r\,dr\,d\theta\\
&=\int_{0}^{\pi/4}(\cos\theta\sin\theta)\,d\theta\int_{0}^{\sqrt{2}} r^2\,dr\\
&=[\sin\theta-\cos\theta]_0^{\pi/4}\left[\frac{1}{3}r^3\right]_0^{\sqrt{2}}\\
&=\left[\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}-0+1\right]\cdot\frac{1}{3}\left(2\sqrt{2}-0\right)\\
&=\frac{2\sqrt{2}}{3}
\end{aligned}$$

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## Applications of Double Integrals

5. Find the mass and center of mass of the lamina that occupies the region $D$ and has the given density function $\rho$.

   (a) $D=\{(x,y)\mid 1\le x\le 3, 1\le y\le 4\}$; $\rho(x,y)=ky^2$

   (b) $D$ is bounded by $y=1-x^2$ and $y=0$; $\rho(x,y)=ky$

(a)

$$\begin{aligned}
m &= \iint_D \rho(x,y)\,dA\\
&=\int_1^3\int_1^4 ky^2\,dy\,dx\\
&=k\int_1^3 dx\int_1^4 y^2\,dy\\
&=k[x]_1^3\left[\frac{1}{3}y^3\right]_1^4\\
&=k(2)(21)=42k,
\end{aligned}$$

$$\begin{aligned}\bar{x} &= \frac{1}{m}\iint\limits_D x\rho(x,y)\,dA \\ &= \frac{1}{42k}\int_1^3\int_1^4 kxy^2\,dy\,dx \\ &= \frac{1}{42}\int_1^3 x\,dx\int_1^4 y^2\,dy \\ &= \frac{1}{42}\left[\frac{1}{2}x^2\right]_1^3\left[\frac{1}{3}y^3\right]_1^4 \\ &= \frac{1}{42}(4)(21) = 2,\end{aligned}$$

$$\begin{aligned}\bar{y} &= \iint Dy\rho(x,y)\,dA \\ &= \frac{1}{42k}\int_1^3\int_1^4 ky^3\,dy\,dx \\ &= \frac{1}{42}\int_1^3 dx\int_1^4 y^3\,dy \\ &= \frac{1}{42}[x]_1^3\left[\frac{1}{4}y^4\right]_1^4 \\ &= \frac{1}{42}(2)\left(\frac{255}{4}\right) = \frac{85}{28}\end{aligned}$$

. Hence $m = 42k$, $(\bar{x},\bar{y}) = (2, \frac{85}{28})$.

(b)

$$\begin{aligned}m &= \int_{-1}^1\int_0^{1-x^2} ky\,dy\,dx \\ &= k\int_{-1}^1\left[\frac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx \\ &= \frac{1}{2}k\int_{-1}^1 (1-x^2)^2\,dx \\ &= \frac{1}{2}k\int_{-1}^1 (1-2x^2+x^4)\,dx \\ &= \frac{1}{2}k\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_{-1}^1 \\ &= \frac{1}{2}k\left(1-\frac{2}{3}+\frac{1}{5}+1-\frac{2}{3}+\frac{1}{5}\right) \\ &= \frac{8}{15}k,\end{aligned}$$

$$\begin{aligned}\bar{y} &= \frac{1}{m}\int_{-1}^1\int_0^{1-x^2} kxy\,dy\,dx \\ &= \frac{k}{m}\int_{-1}^1\left[\frac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx \\ &= \frac{1}{2}\frac{k}{m}\int_{-1}^1 x(1-x^2)^2\,dx \\ &= \frac{1}{2}\frac{k}{m}\int_{-1}^1 (x-2x^3+x^5)\,dx \\ &= \frac{1}{2}\frac{k}{m}\left[\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{1}{6}x^6\right]_{-1}^1 \\ &= \frac{1}{2}\frac{k}{m}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}-\frac{1}{2}+\frac{1}{2}-\frac{1}{6}\right) = 0,\end{aligned}$$

$$\begin{aligned}\bar{x} &= \frac{1}{m}\int_{-1}^1\int_0^{1-x^2} ky^2\,dy\,dx \\ &= \frac{k}{m}\int_{-1}^1\left[\frac{1}{3}y^3\right]_{y=0}^{y=1-x^2} dx \\ &= \frac{1}{3}\frac{k}{m}\int_{-1}^1 (1-x^2)^3\,dx \\ &= \frac{1}{3}\frac{k}{m}\int_{-1}^1 \left(1-3x^2+3x^4-x^6\right)\,dx \\ &= \frac{1}{3}\frac{k}{m}\left[x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7\right]_{-1}^1 \\ &= \frac{1}{3}\frac{k}{m}\left(1-1+\frac{3}{5}-\frac{1}{7}+1-1+\frac{3}{5}-\frac{1}{7}\right) \\ &= \frac{32}{105}\frac{k}{m}.\end{aligned}$$

Hence $m = \dfrac{8}{15}k$,
$(\bar{x},\bar{y}) = \left(0, \dfrac{32k/105}{8k/15}\right) = \left(0, \dfrac{4}{7}\right)$.

6. A lamina occupies the part of the disk $x^2+y^2 \le 1$ in the first quadrant. Find the center of mass if the density at any point is proportional to the distance from the $x$-axis.

$$\rho(x,y) = ky = kr\sin\theta,$$

$$\begin{aligned}m &= \int_0^{\pi/2}\int_0^1 kr^2\sin\theta\,dr\,d\theta \\ &= \frac{1}{3}k\int_0^{\pi/2}\sin\theta\,d\theta \\ &= \frac{1}{3}l[-\cos\theta]_0^{\pi/2} \\ &= \frac{1}{3}k,\end{aligned}$$

$x$-weighted moment

$$\begin{aligned}
&= \int_0^{\pi/2}\int_0^1 kr^2 \sin\theta(r\cos\theta)\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 kr^3 \sin\theta\cos\theta\,dr\,d\theta \\
&= \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \\
&= \frac{1}{8}k[-\cos 2\theta]_0^{\pi/2} \\
&= \frac{1}{8}k,
\end{aligned}$$

$y$-weighted moment

$$\begin{aligned}
&= \int_0^{\pi/2}\int_0^1 kr^2 \sin\theta(r\sin\theta)\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 kr^3 \sin^2\theta\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^1 kr^3 \sin^2\theta\,dr\,d\theta \\
&= \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\,d\theta \\
&= \frac{1}{8}k[\theta + \sin 2\theta]_0^{\pi/2} \\
&= \frac{\pi}{16}k.
\end{aligned}$$

The center of mass then is

$$\begin{aligned}
(\bar{x}, \bar{y}) &= \frac{1}{\text{mass}}\,(x\text{-weighted moment}, y\text{-weighted moment}) \\
&= \left(\frac{3}{k}\right)\left(\frac{k}{8}, \frac{\pi k}{16}\right) \\
&= \left(\frac{3}{8}, \frac{3\pi}{16}\right)
\end{aligned}$$

7. The boundary of a lamina consists of the semicircles $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portions of the $x$-axis that join them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

$$\rho(x, y) = k\sqrt{x^2+y^2} = kr,$$

$$\begin{aligned}
m &= \iint D\rho(x,y)\,dA \\
&= \int_0^{\pi}\int_1^2 kr\cdot r\,dr\,d\theta \\
&= k\int_0^{\pi} d\theta \int_1^2 r^2\,dr \\
&= k(\pi)\left[\frac{1}{3}r^3\right]_1^2 \\
&= \frac{7}{3}\pi k,
\end{aligned}$$

and

$x$-weighted moment

$$\begin{aligned}
&= \iint_D x\rho(x,y)\,dA \\
&= \int_0^{\pi}\int_1^2 (r\cos\theta)(kr)r\,dr\,d\theta \\
&= k\int_0^{\pi}\cos\theta\,d\theta\int_1^2 r^3\,dr \\
&= k[\sin\theta]_0^{\pi}\left[\frac{1}{4}r^4\right]_1^2 \\
&= k(0)\left(\frac{15}{4}\right) = 0.
\end{aligned}$$

This zero value is to be expected, as the region and density function are symmetric about the $y$-axis.

$y$-weighted moment

$$\begin{aligned}
&= \iint_D y\rho(x,y)\,dA \\
&= \int_0^{\pi}\int_1^2 (r\sin\theta)(kr)r\,dr\,d\theta \\
&= k\int_0^{\pi}\sin\theta\,d\theta\int_1^2 r^3\,dr \\
&= k[-\cos\theta]_0^{\pi}\left[\frac{1}{4}r^4\right]_1^2 \\
&= k(1+1)\left(\frac{15}{4}\right) = \frac{15}{2}k.
\end{aligned}$$

Hence $(\bar{x}, \bar{y}) = \left(0, \frac{15k/2}{7\pi k/3}\right) = \left(0, \frac{45}{14\pi}\right)$.