Week #9

Some problems and solutions selected or adapted from Stewart Calculus.

Iterated Integrals

1. Calculate the iterated integral.

   (a) $\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx$

   (b) $\int_0^1 \int_0^1 \nu(u+\nu^2)^4\, du\, d\nu$

(a)

$$\begin{aligned}\int_1^4 \int_0^2 (6x^2y - 2x)\, dy\, dx &= \int_1^4 \left[3x^2y^2 - 2xy\right]_{y=0}^{y=2}\, dx\\ &= \int_1^4 \left(12x^2 - 4x\right)\, dx\\ &= \left[4x^3 - 2x^2\right]_1^4\\ &= (256 - 32) - (4 - 2)\\ &= 222\end{aligned}$$

(b)

$$\begin{aligned}&\int_0^1 \int_0^1 \nu(u+\nu^2)^4\, du\, d\nu\\ &= \int_0^1 \left[\frac{1}{5}\nu(u+\nu^2)^5\right]_{u=0}^{u=1} d\nu\\ &= \frac{1}{5}\int_0^1 \nu\left[\left(1+\nu^2\right)^5 - \left(0+\nu^2\right)^5\right] d\nu\\ &= \frac{1}{5}\int_0^1 \left[\nu\left(1+\nu^2\right)^5 - \nu^{11}\right] d\nu\\ &= \frac{1}{5}\left[\frac{1}{2}\cdot\frac{1}{6}\left(1+\nu^2\right)^6 - \frac{1}{12}\nu^{12}\right]_0^1\\ &\quad \left[\begin{array}{c}\text{substitute } t = 1+v^2\\ \Rightarrow dt = 2v\, dv \text{ in the first term}\end{array}\right]\\ &= \frac{1}{60}\left[\left(2^6 - 1\right) = (1-0)\right]\\ &= \frac{1}{60}(63-1)\\ &= \frac{31}{30}\end{aligned}$$

2. Calculate the double integral.

$$\iint_R \frac{xy^2}{x^2+1}\, dA$$

   where $R = \{(x,y) \mid 0 \le x \le 1, -3 \le y \le 3\}$.

$$\begin{aligned}\iint_R \frac{xy^2}{x^2+1}\, dA &= \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx\\ &= \int_0^1 \frac{x}{x^2+1}\, dx \int_{-3}^3 y^2\, dy\\ &= \left[\frac{1}{2}\ln\left(x^2+1\right)\right]_0^1 \left[\frac{1}{3}y^3\right]_{-3}^3\\ &= \frac{1}{2}(\ln 2 - \ln 1)\cdot\frac{1}{3}(27+27)\\ &= 9\ln 2\end{aligned}$$

3. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x,y) \mid -1 \le x \le 2, -1 \le y \le 1\}$.

The solid lies under the plane $4x + 6y - 2z + 15 = 0$ or $z = 2x + 3y + \frac{15}{2}$ so

$$\begin{aligned}V &= \iint_R \left(2x + 3y + \frac{15}{2}\right) dA\\ &= \int_{-1}^1 \int_{-1}^2 \left(2x + 3y + \frac{15}{2}\right) dx\, dy\\ &= \int_{-1}^1 \left[x^2 + 3xy + \frac{15}{2}x\right]_{x=-1}^{x=2} dy\\ &= \int_{-1}^1 \left[(19+6y) - \left(-\frac{13}{2} - 3y\right)\right] dy\\ &= \int_{-1}^1 \left(\frac{51}{2} + 9y\right) dy\\ &= \left[\frac{51}{2}y + \frac{9}{2}y^2\right]_{-1}^1\\ &= 30 - (-21) = 51\end{aligned}$$

Double Integrals over General Regions

4. Evaluate the double integral.

   (a) $\displaystyle\iint_D y^2 \, dA$,
   $D = \{(x, y) \mid -1 \le y \le 1, -y-2 \le x \le y\}$

   (b) $\displaystyle\iint_D x \, dA$, $D = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le \sin x\}$

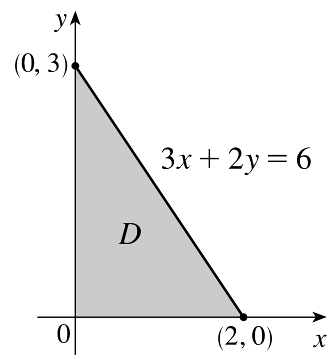
(a)

$$\iint_D y^2 \, dA = \int_{-1}^{1} \int_{-y-2}^{y} y^2 \, dx \, dy$$
$$= \int_{-1}^{1} \left[xy^2\right]_{x=-y-2}^{x=y} \, dy$$
$$= \int_{-1}^{1} y^2[y-(-y-2)] \, dy$$
$$= \int_{-1}^{1} \left(2y^3 + 2y^2\right) \, dy$$
$$= \left[\frac{1}{2}y^4 + \frac{2}{3}y^3\right]_{-1}^{1}$$
$$= \frac{1}{2} + \frac{2}{3} - \frac{1}{2} + \frac{2}{3}$$
$$= \frac{4}{3}$$

(b)

$$\iint_D x \, dA = \int_0^{\pi} \int_0^{\sin x} x \, dy \, dx$$
$$= \int_0^{\pi} [xy]_{y=0}^{y=\sin x} \, dx$$
$$= \int_0^{\pi} x \sin x \, dx$$
$$\left[\begin{array}{c}\text{integrate by parts} \\ \text{with } u = x,\ dv = \sin x \, dx\end{array}\right]$$
$$= [-x\cos x + \sin x]_0^{\pi}$$
$$= -\pi\cos\pi + \sin\pi + 0 - \sin 0$$
$$= \pi$$

5. Find the volume of the solid bounded by the coordinate planes and the plane $3x + 2y + z = 6$.



$$V = \int_0^2 \int_0^{2-\frac{3}{2}x} (6 - 3x - 2y) \, dy \, dx$$
$$= \int_0^2 \left[6y - 3xy - y^2\right]_{y=0}^{y=3=\frac{3}{2}x} \, dx$$
$$= \int_0^2 \left[6\left(3 - \frac{3}{2}x\right) - 3x\left(3 - \frac{3}{2}x\right) - \left(3 - \frac{3}{2}x\right)^2\right] \, dx$$
$$= \int_0^2 \left(\frac{9}{4}x^2 - 9x + 9\right) \, dx$$
$$= \left[\frac{3}{4}x^3 - \frac{9}{2}x^2 + 9x\right]_0^2$$
$$= 6 - 0 = 6$$

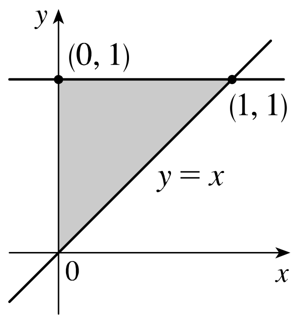
6. Sketch the region of integration and change the order of integration.

   (a) $\displaystyle\int_0^1 \int_0^y f(x, y) \, dx \, dy$

   (b) $\displaystyle\int_1^2 \int_0^{\ln x} f(x, y) \, dy \, dx$

(a) Because the region of integration is

$$D = \{(x, y) \mid 0 \le x \le y, 0 \le y \le 1\}$$
$$= \{(x, y) \mid x \le y \le 1, 0 \le x \le 1\}$$



we have

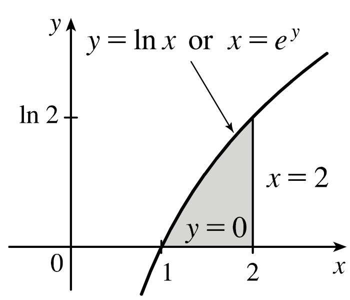
$$\int_0^{\pi/2}\int_0^{\cos x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^1\int_x^1 f(x,y)\,dy\,dx$$

(b) Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2\int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\ln 2}\int_{e^y}^2 f(x,y)\,dx\,dy$$



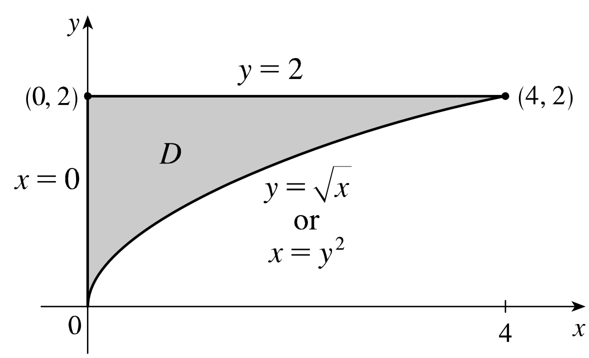
7. Evaluate the integral by reversing the order of integration.

(a) $\displaystyle\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx$

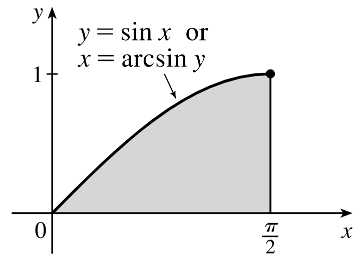
(b) $\displaystyle\int_0^1\int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$

(a)

$$\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx = \int_0^2\int_0^{y^2}\frac{1}{y^3+1}\,dx\,dy$$
$$= \int_0^2 \frac{1}{y^3+1}[x]_{x=0}^{x=y^2}\,dy$$
$$= \int_0^2 \frac{y^2}{y^3+1}\,dy$$
$$= \frac{1}{3}\left[\ln\left|y^3+1\right|\right]_0^2$$
$$= \frac{1}{3}(\ln 9 - \ln 1)$$
$$= \frac{1}{3}\ln 9$$



(b)



$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dx\,dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\,dy\ \left[y\right]_{y=0}^{y=\sin x}\ dx$$
$$= \int_0^{\pi/2} \cos x\sqrt{1+\cos^2 x}\sin x\,dx$$
$$\left[\begin{array}{c}\text{Let } u=\cos x,\ du=-\sin x\,dx,\\ dx=du/(-\sin x)\end{array}\right]$$
$$= \int_1^0 -u\sqrt{1+u^2}\,du$$
$$= -\frac{1}{3}\left[\left(1+u^2\right)^{3/2}\right]_1^0$$
$$= \frac{1}{3}\left(\sqrt{8}-1\right)$$
$$= \frac{1}{3}\left(2\sqrt{2}-1\right)$$