Week #8

Some problems and solutions selected or adapted from Stewart Calculus.

Taylor and Maclaurin Series

1. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

   (a) $f(x) = (1-x)^{-2}$

   (b) $f(x) = 2^x$

(a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1-x)^{-2}$	1
1	$(-2)(1-x)^{-3}(-1)$	
	$= 2(1-x)^{-3}$	2
2	$(2)(3)(1-x)^{-4}$	3!
3	$(2)(3)(4)(1-x)^{-5}$	4!
4	$(2)(3)(4)(5)(1-x)^{-6}$	5!
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
(1-x)^{-2} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\
&\quad + \frac{f^{(4)}(0)}{4!}x + \cdots \\
&= 1 + 2x + \frac{6}{2}x^2 + \frac{24}{6}x^3 + \frac{120}{24}x^4 + \cdots \\
&= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots \\
&= \sum_{n=0}^{\infty} (n+1)x^n
\end{aligned}$$

(b)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	2^x	1
1	$2^x(\ln 2)$	$\ln 2$
2	$2^x(\ln 2)^2$	$(\ln 2)^2$
3	$2^x(\ln 2)^3$	$(\ln 2)^3$
4	$2^x(\ln 2)^4$	$(\ln 2)^4$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
2^x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\
&= \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}x^n
\end{aligned}$$

2. Find the Taylor series for $f(x)$ centered at the given value of a. [Asume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

   (a) $f(x) = x^4 - 3x^2 + 1$, $a = 1$

   (b) $f(x) = \ln x$, $a = 2$

(a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^4 - 3x^2 + 1$	-1
1	$4x^3 - 6x$	-2
2	$12x^2 - 6$	6
3	$24x$	24
4	24	24
5	0	0
6	0	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(x) = 0$ for $n \geq 5$, so f has a finite series expansion about $a = 1$.

$$\begin{aligned}
f(x) &= x^4 - 3x^2 + 1 \\
&= \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n \\
&= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 \\
&\quad + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\
&= -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 \\
&\quad + (x-1)^4
\end{aligned}$$

(b)

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	$1/x$	$1/2$
2	$-1/x^2$	$-1/2^2$
3	$2/x^3$	$2/2^3$
4	$-6/x^4$	$-6/2^4$
5	$24/x^5$	$24/2^5$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}
f(x) &= \ln x \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \frac{\ln 2}{0!}(x-2)^0 + \frac{1}{1!2^1}(x-2)^1 + \frac{-1}{2!2^2}(x-2)^2 \\
&\quad + \frac{2}{3!2^3}(x-2)^3 + \frac{-6}{4!2^4}(x-2)^4 \\
&\quad + \frac{24}{5!2^5}(x-2)^5 + \cdots \\
&= \ln 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n-1)!}{n!2^n}(x-2)^n \\
&= \ln 2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}(x-2)^n
\end{aligned}$$

3. Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the function.

$$f(x) = \frac{x}{\sqrt{4+x^2}}$$

The function $\frac{x}{\sqrt{4+x^2}}$ is closest in form to the binomial expansion,
$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 = \frac{k(k-1)(k-2)}{3!}x^3 + \ldots$
just with an extra x multiplier and some fiddling to match the $(4+x^2)$ to the binomial's $(1+x)$ form.
We start by rewriting using a power $k = \frac{-1}{2}$:

$$\frac{x}{\sqrt{4+x^2}} = x(4+x^2)^{-1/2}$$

Next we must write the binomial in the form $(1 + \text{expression})$, so we'll factor out a 4.

$$x(4+x^2)^{-1/2} = x\left[4^{-1/2}\left(1+\frac{x^2}{4}\right)^{-1/2}\right]$$

$$\text{tidying:} \quad = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2}$$

We can now use the binomial expansion, working with $x \to \frac{x^2}{4}$ and $k = \frac{-1}{2}$.

$$\frac{x}{2}\left[1+\frac{x^2}{4}\right]^{-1/2} = \frac{x}{2}\left[1 + \frac{-1}{2}\left(\frac{x^2}{4}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \ldots\right]$$

$$\text{tidying signs, powers:} \quad = \frac{x}{2}\left[1 + (-1)\left(\frac{1}{2}\right)\left(\frac{x^2}{4}\right) + \frac{\left(\frac{(1)(3)}{2^2}\right)}{2!}\left(\frac{x^4}{4^2}\right) + (-1)\frac{\left(\frac{(1)(3)(5)}{2^3}\right)}{3!}\left(\frac{x^6}{4^3}\right) + \ldots\right]$$

$$\text{grouping the } 4{=}2^2 \text{ and 2 powers:} \quad = \frac{x}{2}\left[1 + (-1)\left(\frac{1}{2^3}\right)x^2 + \left(\frac{(1)(3)}{2^6}\right)\frac{1}{2!}x^4 + (-1)\left(\frac{(1)(3)(5)}{2^9}\right)\frac{1}{3!}x^6 + \ldots\right]$$

Putting this in summation notation, it is difficult to fit that first term (1) into the pattern of the later terms, so we break it off from the rest of the sum:

$$\frac{x}{2}\left[1+\frac{x^2}{4}\right]^{-1/2} = \frac{x}{2}\left[\underbrace{1}_{\text{skip}} + \underbrace{(-1)\left(\frac{1}{2^3}\right)x^2}_{n=1} + \underbrace{\left(\frac{(1)(3)}{2^6}\right)\frac{1}{2!}x^4}_{n=2} + \underbrace{(-1)\left(\frac{(1)(3)(5)}{2^9}\right)\frac{1}{3!}x^6}_{n=3} + \ldots\right]$$

$$= \frac{x}{2}\left(1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^{3n}\ n!}x^{2n}\right)$$

Finally we can multiply in the the $\frac{x}{2}$:

$$\frac{x}{2}\left(1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^{3n}\ n!}x^{2n}\right) = \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^{3n+1}\ n!}x^{2n+1}$$

4. Use the Maclaurin series for $\cos x$ to compute $\cos 5^\circ$ correct to five decimal places.

$$5^\circ = 5^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{36} \text{ radians}$$

and

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

so

$$\cos\frac{\pi}{36} = 1 - \frac{(\pi/36)^2}{2!} + \frac{(\pi/36)^4}{4!} - \frac{(\pi/36)^6}{6!} + \cdots .$$

Now, $1 - \dfrac{(\pi/36)^2}{2!} \approx 0.99619$ and adding $\dfrac{(\pi/36)^4}{4!} \approx 2.4 \times 10^{-6}$ does not affect the fifth decimal place, so $\cos 5^\circ \approx 0.99619$ by the Alternating Series Estimation Theorem.

5. Use series to approximate the definite integral to within the indicated accuracy.

   (a) $\displaystyle\int_0^{1/2} x^3 \arctan x \, dx$ (four decimal places)

   (b) $\displaystyle\int_0^{0.4} \sqrt{1+x^4} \, dx$ ($|\text{error}| < 5 \times 10^{-6}$)

(a)

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$, so

$$x^3 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{2n+1}$$

for $|x| < 1$ and

$$\int x^3 \arctan x \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+5}}{(2n+1)(2n+5)}.$$

Since $\frac{1}{2} < 1$, we have

$$\begin{aligned}\int_0^{1/2} x^3 \arctan x \, dx &= \sum_{n=0}^{\infty} \frac{(1/2)^{2n+5}}{(2n+1)(2n+5)} \\ &= \frac{(1/2)^5}{1/cdot5} - \frac{(1/2)^7}{3 \cdot 7} + \frac{(1/2)^9}{5 \cdot 9} \\ &\quad - \frac{(1/2)^{11}}{7 \cdot 11} + \cdots .\end{aligned}$$

Now

$$\frac{(1/2)^5}{1 \cdot 5} - \frac{(1/2)^7}{(3 \cdot 7)} + \frac{(1/2)^9}{5 \cdot 9} \approx 0.0059$$

and subtracting

$$\frac{(1/2)^{11}}{7 \cdot 11} \approx 6.3 \times 10^{-6}$$

does not affect the fourth decimal place, so $\displaystyle\int_0^{1/2} x^3 \arctan x \, dx \approx 0.0059$ by the Alternating Series Estimation Theorem.

(b)

$$\begin{aligned}\sqrt{1+x^4} &= (1+x^4)^{1/2} \\ &= \sum_{n=0}^{\infty} \binom{1/2}{n} (x^4)^n,\end{aligned}$$

so

$$\int \sqrt{1+x^4} \, dx = C + \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{x^{4n+1}}{4n+1}$$

and hence, since $0.4 < 1$, we have

$$\begin{aligned} I &= \int_0^{0.4} \sqrt{1+x^4}\, dx \\ &= \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{(0.4)^{4n+1}}{4n+1} \\ &= (1)\frac{(0.4)^1}{0!} + \frac{\frac{1}{2}}{1!}\frac{(0.4)^5}{5} + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}\frac{(0.4)^9}{9} \\ &\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}\frac{(0.4)^{13}}{13} \\ &\quad + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}\frac{(0.4)^{17}}{17} + \cdots \\ &= 0.4 + \frac{(0.4)^5}{10} - \frac{(0.4)^9}{72} + \frac{(0.4)^{13}}{208} \\ &\quad - \frac{5(0.4)^{17}}{2176} + \cdots \end{aligned}$$

Now

$$\frac{(0.4)^9}{72} \approx 3.6 \times 10^{-6} < 5 \times 10^{-6},$$

so by the Alternating Series Estimation Theorem,

$$I \approx 0.4 + \frac{(0.4)^5}{10} \approx 0.40102$$

(correct to five decimal places).

Applications of Taylor Polynomials

6. Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \cos x$ centered at $x = \frac{\pi}{2}$. Graph f and T_3 on the same screen.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\cos x$	0
1	$-\sin(x)$	-1
2	$-\cos(x)$	0
3	$\sin(x)$	1

$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(\pi/2)}{n!}\left(x - \frac{\pi}{2}\right)^n \\ &= -\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^n \end{aligned}$$

7. (a) Approximate $f(x) = e^{x^2}$ by a Taylor polynomial with degree 3 at $x = 0$.
   (b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $0 \le x \le 0.1$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

(a) $f(x) = e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \le \frac{M}{4!}|x|^4$, where $|f^{(4)}(x)| \le M$. Now $0 \le x \le 0.1 \Rightarrow x^4 \le (0.1)^4$, and letting $x = 0.1$ gives

$$\begin{aligned} |R_3(x)| &\le \frac{e^{0.01}(12 + 0.48 + 0.0016)}{24}(0.1)^4 \\ &\approx 0.00006. \end{aligned}$$

8. Use the information from Exercise 6 to estimate $\cos(80^\circ)$ correct to five decimal places.

From exercise 6,

$$\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3 + R_3(x),$$

where

$$|R_3(x)| \le \frac{M}{4!}|x - \frac{\pi}{2}|^4$$

with

$$|f^{(4)}(x)| = |\cos x| \le M = 1.$$

Now

$$x = 80^\circ = (90^\circ - 10^\circ) = (\frac{\pi}{2} - \frac{\pi}{18}) = \frac{4\pi}{9} \text{ radians,}$$

so the error is

$$|R_3\left(\frac{4\pi}{9}\right)| \le \frac{1}{24}\left(\frac{\pi}{18}\right)^4 \approx 0.000039,$$

which means our estimate would *not* be accurate to five decimal places. However, $T_3 = T_4$, so we can use

$$|R_4\left(\frac{4\pi}{9}\right)| \le \frac{1}{120}\left(\frac{\pi}{18}\right)^5 \approx 0.000001.$$

Therefore, to five decimal places,

$$\cos 80\,\text{degree} \approx -\left(-\frac{\pi}{18}\right) + \frac{1}{6}\left(-\frac{\pi}{18}\right)^3 \approx 0.17365.$$

9. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

All derivatives of e^x are e^x, so

$$|R_n(x)| \le \frac{e^x}{(n+1)!}|x|^{n+1},$$

where $0 < x < 0.1$. Letting $x = 0.1$,

$$R_n(0.1) \le \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001,$$

and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2, \text{and}, 3$, we can estimte $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.1051\bar{6}$ and $e^{0.1} \approx 1.10517$.)

10. Use the Alternating Series Estimation Theorem or Taylor's Inequality to estimate the range of values of x for which the given approximation is accurate to within the stated error. Check your answer graphically.
    (a) $\sin x \approx x - \frac{x^3}{6}$ ($|error| < 0.01$)
    (b) $\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}$ ($|error| < 0.05$)

(a)

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

By the Alternating Series Estimation Theorem, the error in the approximation

$$\sin x = x - \frac{1}{3!}x^3$$

is less than $|\frac{1}{5!}x^5| < 0.01 \Leftrightarrow |x^5| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ interesect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate for only $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.

(b)

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

By the Alternating Series Estimation Theorem, the error is less than $|-\frac{1}{7}x^7| < 0.05 \Leftrightarrow$
$|x^7| < 0.35 \Leftrightarrow |x| < (0.35)^{1/7} \approx 0.8607$. The curves $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ and $y = \arctan x + 0.05$ intersect at $x \approx 0.9245$, so the graph confirms our estimate. Since both the arctangent function and the given approximation are odd functions, we need to check the estimate for only $x > 0$. Thus, the desired range of values for x is $-0.86 < x < 0.86$.