

## Week #7

Some problems and solutions selected or adapted from Stewart Calculus.

### Series

1. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

(a)  $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$

(b)  $10 - 2 + 0.4 - 0.08 + \dots$

(c)  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

(d)  $\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

- (a)  $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$  is a geometric series with ratio  $r = -\frac{4}{3}$ . Since  $|r| = \frac{4}{3} > 1$ , the series diverges.

- (b)  $10 - 2 + 0.4 - 0.08 + \dots$  is a geometric series with ratio  $-\frac{2}{10} = -\frac{1}{5}$ . Since  $|r| = \frac{1}{5} < 1$ , the series converges to

$$\frac{a}{1-r} = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}.$$

- (c)

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} - \frac{3}{4^2} + \frac{3^2}{4^3} - \frac{3^3}{4^4} + \dots$$

The series is geometric with  $a = \frac{1}{4}$  and ratio  $r = -\frac{3}{4}$ . Since  $|r| = \frac{3}{4} < 1$ , it converges to

$$\frac{a}{1-r} = \frac{\frac{1}{4}}{1-(-3/4)} = \frac{1}{7}.$$

- (d)

$$\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{\pi}{3}\right)^n$$

is a geometric series with ratio  $r = \frac{\pi}{3}$ . Since  $|r| \approx \frac{3.1415}{3} > 1$ , the series diverges.

2. Find the values of  $x$  for which the series converges. Find the sum of the series for those values of  $x$ .

(a)  $\sum_{n=1}^{\infty} (-5)^n x^n$

(b)  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

- (a)

$$\sum_{n=1}^{\infty} (-5)^n x^n = \sum_{n=1}^{\infty} (-5x)^n$$

is a geometric series with  $r = -5x$ , so the series converges  $\Leftrightarrow |r| < 1 \Leftrightarrow |-5x| < 1 \Leftrightarrow |x| < \frac{1}{5}$ , that is,  $-\frac{1}{5} < x < \frac{1}{5}$ .

For  $x$  values in this range, the sum of the series can be computed using the geometric series formula with first term  $n = 1$ , giving  $a = -5x$ , and the common ratio between terms of  $r = -5x$  too:

$$\frac{a}{1-r} = \frac{-5x}{1-(-5x)} = \frac{-5x}{1+5x}.$$

- (b)

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

is a geometric series with  $r = \frac{x}{2}$ , so the series converges  $\Leftrightarrow |r| < 1 \Leftrightarrow \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2 \Leftrightarrow -2 < x < 2$ . In that case, the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1-x/2}$$

## Alternating Series

3. Test the series for convergence or divergence.

- (a)  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$   
 (b)  $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$   
 (c)  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$   
 (d)  $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

(a)

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} b_n. \end{aligned}$$

Now  $b_n = \frac{1}{2n+1} > 0$ ,  $\{b_n\}$  is decreasing, and  $\lim_{n \rightarrow \infty} b_n = 0$ , so the series converges by the Alternating Series Test.

(b)

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} \\ &= \sum_{n=1}^{\infty} (-1)^n b_n. \end{aligned}$$

Now

$$\begin{aligned} \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} \frac{3-1/n}{2+1/n} \\ &= \frac{3}{2} \neq 0. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} a_n \neq 0$  (in fact the limit does not exist), the series diverges by the Test for Divergence.

(c)

$$b_n = \frac{n^2}{n^3+4} > 0$$

for all  $n \geq 1$ .  $\{b_n\}$  is decreasing for  $n \geq 2$  since

$$\begin{aligned} \left( \frac{x^2}{x^3+4} \right)' &= \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} \\ &= \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} \\ &= \frac{x(8-x^3)}{(x^3+4)^2} < 0 \end{aligned}$$

for  $x > 2$ . Also,

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1/n}{1+4/n^3} = 0.$$

Thus, the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$  converges by the Alternating Series Test.

(d)

$$\begin{aligned} \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} e^{2/n} \\ &= e^0 = 1, \end{aligned}$$

so  $\lim_{n \rightarrow \infty} (-1)^{n-1} e^{2/n}$  does not exist. Thus, the series  $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$  diverges by the Test for Divergence.

4. Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

- (a)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$  ( $|\text{error}| < 0.00005$ )  
 (b)  $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$  ( $|\text{error}| < 0.000005$ )

(a) The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$  satisfies (i) of the Alternating Series test because

$$\frac{1}{(n+1)^6} < \frac{1}{n^6}$$

and (ii)

$$\lim_{n \rightarrow \infty} \frac{1}{n^6} = 0,$$

so the series is convergent. Now

$$\begin{aligned} b_5 &= \frac{1}{5^6} \\ &= 0.000064 > 0.00005 \end{aligned}$$

and

$$\begin{aligned} b_6 &= \frac{1}{6^6} \\ &\approx 0.00002 < 0.00005, \end{aligned}$$

so by the Alternating Series Estimation Theorem,  $n = 5$ . (That is, since the 6th term is less than the desired error, we need to add the first five terms to get the sum to the desired accuracy.)

(b) The series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$  satisfies (i) of the Alternating Series Test because

$$\frac{1}{10^{n+1}(n+1)!} < \frac{1}{10^n n!}$$

and (ii)

$$\lim_{n \rightarrow \infty} \frac{1}{10^n n!} = 0,$$

so the series is convergent. Now

$$b_3 = \frac{1}{10^3 3!} \approx 0.000167 > 0.000005$$

and

$$b_4 = \frac{1}{10^4 4!} = 0.000004 < 0.000005,$$

so by the Alternating Series Estimation Theorem,  $n = 4$  (since the series starts with  $n = 0$ , not  $n = 1$ ). (That is, since the 5th term is less than

the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

5. Is the 50th partial sum  $s_{50}$  of the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  an overestimate or underestimate of the total sum? Explain.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ &\quad + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots \end{aligned}$$

The 50th partial sum of this series is an underestimate, since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left( \frac{1}{51} - \frac{1}{52} \right) + \left( \frac{1}{53} - \frac{1}{54} \right) + \cdots,$$

and the terms in parentheses are all positive.

## Representations of Functions as Power Series

6. Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{x}{9 + x^2}$$

$$\begin{aligned} f(x) &= \frac{x}{9 + x^2} \\ &= \frac{x}{9} \left[ \frac{1}{1 + (x/3)^2} \right] \\ &= \frac{x}{9} \left[ \frac{1}{1 - \{-(x/3)^2\}} \right] \\ &= \frac{x}{9} \sum_{n=0}^{\infty} \left[ -\left(\frac{x}{3}\right)^2 \right]^n \\ &= \frac{x}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{9^n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}} \end{aligned}$$

The geometric series

$$\sum_{n=0}^{\infty} \left[ -\left(\frac{x}{3}\right)^2 \right]^n$$

converges when

$$\begin{aligned} \left| -\left(\frac{x}{3}\right)^2 \right| &< 1 \\ \Leftrightarrow \frac{|x^2|}{9} &< 1 \\ \Leftrightarrow |x|^2 &< 9 \\ \Leftrightarrow |x| &< 3, \end{aligned}$$

so the radius of convergence is  $R = 3$  and the interval of convergence is  $I = (-3, 3)$  or  $x \in (-3, 3)$ .

7. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

(a)

$$\begin{aligned}
f(x) &= \frac{1}{(1+x)^2} \\
&= \frac{d}{dx} \left( \frac{-1}{1+x} \right) \\
&= -\frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \\
&= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n
\end{aligned}$$

with  $R = 1$ .

(b)

$$\begin{aligned}
f(x) &= \frac{1}{(1+x)^3} \\
&= -\frac{1}{2} \frac{d}{dx} \left[ \frac{1}{(1+x)^2} \right] \\
&= -\frac{1}{2} \frac{d}{dx} \left[ \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \right] \\
&= -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) n x^{n-1} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n
\end{aligned}$$

with  $R = 1$ .

(c)

$$\begin{aligned}
f(x) &= \frac{x^2}{(1+x)^3} \\
&= x^2 \cdot \frac{1}{(1+x)^3} \\
&= x^2 \cdot \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n \\
&= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^{n+2}
\end{aligned}$$

To write the power series with  $x^n$  rather than  $x^{n+2}$ , we will *decrease* each occurrence of  $n$  in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us

$$\frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n)(n-1) x^n$$

with  $R = 1$ .

8. Find a power series representation for the function and determine the radius of convergence.

(a)  $f(x) = \ln(5-x)$

(b)  $f(x) = \frac{x}{(1+4x)^2}$

(a)

$$\begin{aligned}
f(x) &= \ln(5-x) \\
&= -\int \frac{dx}{5-x} \\
&= -\frac{1}{5} \int \frac{dx}{1-x/5} \\
&= -\frac{1}{5} \int \left[ \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n \right] dx \\
&= C - \frac{1}{5} \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)} \\
&= C - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}
\end{aligned}$$

Putting  $x = 0$ , we get  $C = \ln 5$ . The series converges for  $|x/5| < 1 \Leftrightarrow |x| < 5$ , so  $R = 5$ .

(b) We know that

$$\begin{aligned}
\frac{1}{1+4x} &= \frac{1}{1-(-4x)} \\
&= \sum_{n=0}^{\infty} (-4x)^n.
\end{aligned}$$

Differentiating, we get

$$\begin{aligned}
\frac{-4}{(1+4x)^2} &= \sum_{n=1}^{\infty} (-4)^n n x^{n-1} \\
&= \sum_{n=0}^{\infty} (-4)^{n+1} (n+1) x^n, \text{ so}
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{x}{(1+4x)^2} \\
&= \frac{-x}{4} \cdot \frac{-4}{(1+4x)^2} \\
&= \frac{-x}{4} \sum_{n=0}^{\infty} (-4)^{n+1} (n+1) x^n \\
&= \sum_{n=0}^{\infty} (-1)^n 4^n (n+1) x^{n+1}
\end{aligned}$$

for  $|-4x| < 1 \Leftrightarrow |x| < \frac{1}{4}$ , so  $R = \frac{1}{4}$ .

9. Evaluate the indefinite integral as a power series. What is the radius of convergence?

(a)  $\int \frac{t}{1-t^3} dt$

(b)  $\int x^2 \ln(1+x) dx$

(a) The function  $\frac{t}{1-t^3}$  is similar to the sum of a geometric series, with an additional  $t$  multiplier and a bit of substitution required in the denominator.

$$\begin{aligned} \frac{t}{1-t^3} &= t \left( \frac{1}{1-(t^3)} \right) \quad \text{geometric series w/ } r = t^3 \\ &= t(1+t^3+t^6+t^9+\dots) \\ &= t+t^4+t^7+t^{10}+\dots \\ &= \sum_{n=0}^{\infty} t^{1+3n} \end{aligned}$$

The integral of the original function is equal to the integral of the series, at least on the interval of convergence.

$$\begin{aligned} \int \frac{t}{1-t^3} dx &= \int (t+t^4+t^7+t^{10}+\dots) dx \\ &= \left( \frac{t^2}{2} + \frac{t^5}{5} + \frac{t^8}{8} + \frac{t^{11}}{11} + \dots \right) + C \\ &= C + \sum_{n=0}^{\infty} \frac{t^{2+3n}}{2+3n} \end{aligned}$$

The interval of convergence is defined by the geometric series constraint  $|r| < 1$ , or here  $|t^3| < 1$ , which simplifies to  $|t| < 1$ , or a radius of convergence of  $R = 1$ .

(b)

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

for  $|x| < 1$ , so

$$x^2 \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+2}}{n} \text{ and}$$

$$\int x^2 \ln(1+x) dx = C + \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+3}}{n(n+3)}.$$

$R = 1$  for the series for  $\ln(1+x)$ , so  $R = 1$  for the series representing  $x^2 \ln(1+x)$  as well. The series for  $\int x^2 \ln(1+x) dx$  then also has  $R = 1$ .

10. Use a power series to approximate the definite integral below to six decimal places.

$$\int_0^{0.2} \frac{1}{1+x^5} dx$$

$$\begin{aligned} \frac{1}{1+x^5} &= \frac{1}{1-(-x^5)^n} \\ &= \sum_{n=0}^{\infty} (-x^5)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{5n} \Rightarrow \end{aligned}$$

$$\begin{aligned} \int \frac{1}{1+x^5} dx &= \int \sum_{n=0}^{\infty} (-1)^n x^{5n} dx \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n+1}}{5n+1}. \end{aligned}$$

Thus,

$$\begin{aligned} I &= \int_0^{0.2} \frac{1}{1+x^5} dx \\ &= \left[ x - \frac{x^6}{6} + \frac{x^{11}}{11} - \dots \right]_0^{0.2} \\ &= 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \dots \end{aligned}$$

The series is alternating, so if we use the first two terms, the error is at most  $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$ . So  $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$  to six decimal places.