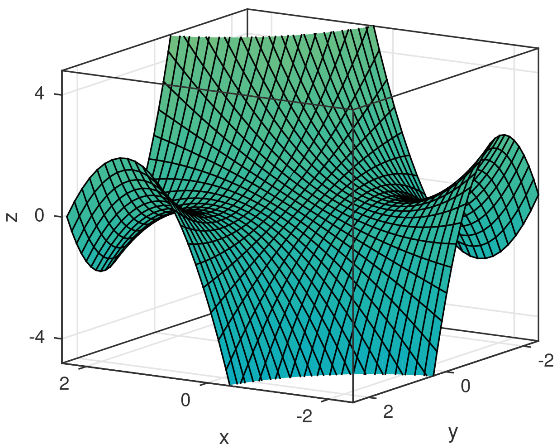
Week #5

Some problems and solutions selected or adapted from Stewart Calculus.

Maximum and Minimum Values

For questions **1-4**, find the local maximum and minimum values and saddle point(s) of the function. If you have access to three-dimensional graphing software (e.g. MATLAB or Wolfram Alpha), graph the function with a domain and viewpoint that reveal all the important aspects of the function.

1. $f(x, y) = (x - y)(1 - xy)$



$$\begin{aligned} f(x, y) &= (x - y)(1 - xy) \\ &= x - y - x^2y + xy^2 \end{aligned}$$

$\Rightarrow f_x = 1 - 2xy + y^2$, $f_y = -1 - x^2 + 2xy$,
$f_{xx} = -2y$, $f_{xy} = -2x + 2y$, $f_{yy} = 2x$.
Then $f_x = 0$ implies $1 - 2xy + y^2 = 0$ and $f_y = 0$ implies $-1 - x^2 + 2xy = 0$.
Adding the two equations gives $1 + y^2 - 1 - x^2 = 0$ $\Rightarrow y^2 = x^2 \Rightarrow y = \pm x$, but if $y = -x$ then $f_x = 0$ implies $1 + 2x^2 + x^2 = 0 \Rightarrow 3x^2 = -1$ which has no real solution.
If $y = x$ then substitution into $f_x = 0$ gives
$1 - 2x^2 + x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x \pm 1$,
so the critical points are $(1, 1)$ and $(-1, -1)$.

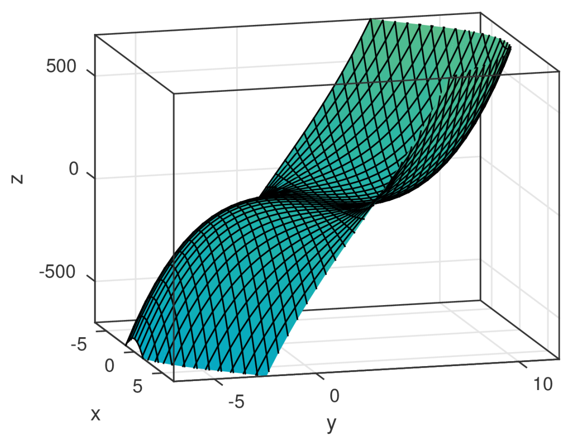
To classify these point: $\begin{aligned} D(1, 1) &= (-2)(2) - 0^2 \\ &= -4 < 0 \end{aligned}$

and

$$\begin{aligned} D(-1, -1) &= (2)(-2) - 0^2 \\ &= -4 < 0, \end{aligned}$$

so $(1, 1)$ and $(-1, -1)$ are saddle points.

2. $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$



$f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2 \Rightarrow$
$f_x = 6xy - 12x$, $f_y = 3y^2 + 3x^2 - 12y$, and
$f_{xx} = 6y - 12$, $f_{xy} = 6x$, $f_{yy} = 6y - 12$.
Then $f_x = 0$ implies $6x(y - 2) = 0$ or $y = 2$.
If $x = 0$ then substitution into $f_y = 0$ gives
$3y^2 - 12y = 0 \Rightarrow 3y(y - 4) = 0 \Rightarrow y = 0$ or $y = 4$, so we have critical points $(0, 0)$ and $(0, 4)$.
If $y = 2$, substitution into $f_y = 0$ gives
$12 + 3x^2 - 24 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$, so we have critical points $(\pm 2, 2)$.

$$\begin{aligned} D(0, 0) &= (-12)(-12) - 0^2 \\ &= 144 > 0 \end{aligned}$$

and $f_{xx}(0, 0) = 12 < 0$, so $f(0, 0) = 2$ is a local maximum.

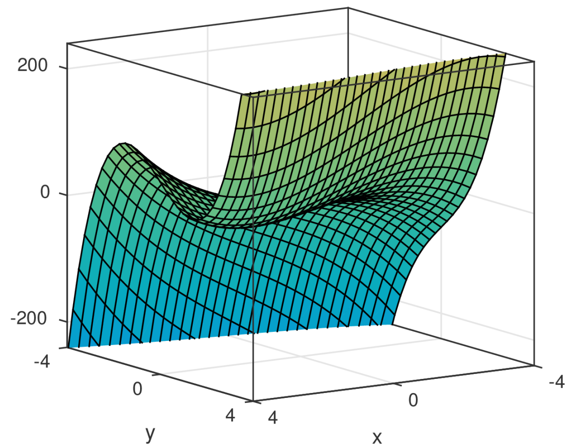
$$\begin{aligned} D(0, 4) &= (12)(12) - 0^2 \\ &= 144 > 0 \end{aligned}$$

and $f_{xx}(0, 4) = 12 > 0$, so $f(0, 4) = -30$ is a local minimum.

$$\begin{aligned} D(2, 2) &= (0)(0) - (\pm 12)^2 \\ &= -144 < 0, \text{ and} \\ D(-2, 2) &= (0)(0) - (\pm - 12)^2 \\ \text{also} &= -144 < 0, \end{aligned}$$

so $(2, 2)$ and $(-2, 2)$ are both saddle points.

3. $f(x, y) = x^3 - 12xy + 8y^3$



Taking partial derivatives, $f_x = 3x^2 - 12y$, $f_y = -12x + 24y^2$, and
$f_{xx} = 6x$, $f_{xy} = -12$, $f_{yy} = 48y$.
Then $f_x = 0$ implies $x^2 = 4y$ and $f_y = 0$ implies $x = 2y^2$.
Substituting the second equation into the first gives $(2y^2)^2 = 4y \Rightarrow 4y^4 = 4y \Rightarrow 4y(y^3 - 1) = 0 \Rightarrow y = 0$ or $y = 1$.
If $y = 0$ then $x = 0$ and if $y = 1$ then $x = 2$, so the critical points are $(0, 0)$ and $(2, 1)$.

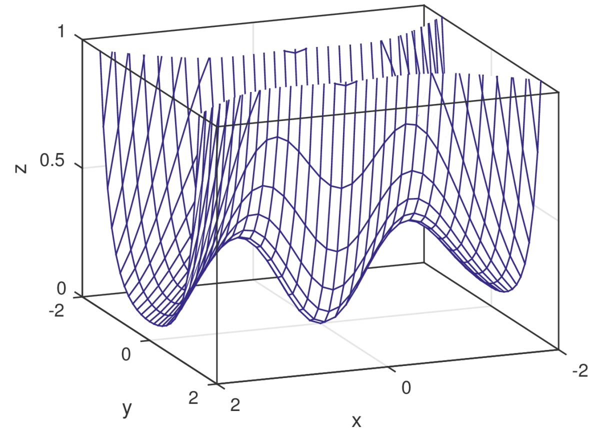
$$\begin{aligned} D(0,0) &= (0)(0) - (-12)^2 \\ &= -144 < 0, \end{aligned}$$

so $(0, 0)$ is a saddle point.

$$\begin{aligned} D(2,1) &= (12)(48) - (-12)^2 \\ &= 432 > 0 \end{aligned}$$

and $f_{xx}(2, 1) = 12 > 0$ so $f(2, 1) = -8$ is a local minimum.

4. $f(x, y) = (x^2 + y^2)e^{y^2 - x^2}$



Taking partial derivatives,

$$\begin{aligned} f_x &= (x^2 + y^2)e^{y^2-x^2}(-2x) + 2xe^{y^2-x^2} \\ &= 2xe^{y^2-x^2}(1 - x^2 - y^2), \end{aligned}$$

$$\begin{aligned} f_y &= (x^2 + y^2)e^{y^2-x^2}(2y) + 2ye^{y^2-x^2} \\ &= 2ye^{y^2-x^2}(1 + x^2 + y^2), \end{aligned}$$

$$\begin{aligned} f_{xx} &= 2xe^{y^2-x^2}(-2x) + (1 - x^2 - y^2) \\ &\quad \times \left(2x\left(-2xe^{y^2-x^2}\right) + 2e^{y^2-x^2}\right) \\ &= 2e^{y^2-x^2}((1 - x^2 - y^2)(1 - 2x^2) - 2x^2), \end{aligned}$$

$$\begin{aligned} f_{xy} &= 2xe^{y^2-x^2}(-2y) + 2x(2y)e^{y^2-x^2}(1 - x^2 - y^2) \\ &= -4xye^{y^2-x^2}(x^2 + y^2), \end{aligned}$$

$$\begin{aligned} f_{yy} &= 2ye^{y^2-x^2}(2y) + (1 + x^2 + y^2) \\ &\quad \times \left(2y\left(2ye^{y^2-x^2}\right) + 2e^{y^2-x^2}\right) \\ &= 2e^{y^2-x^2}((1 + x^2 + y^2)(1 + 2y^2) + 2y^2). \end{aligned}$$

$f_y = 0$ implies $y = 0$, and substituting into $f_x = 0$ gives $2xe^{-x^2}(1 - x^2) = 0 \Rightarrow x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$ and $(\pm 1, 0)$. Now

$$\begin{aligned} D(0,0) &= (2)(2) - 0 \\ &= 4 > 0 \end{aligned}$$

$f_{xx}(0, 0) = 2 > 0$, so $f(0, 0) = 0$ is a local minimum. $D(\pm 1, 0) = (-4e^{-1})(4e^{-1}) - 0 < 0$ so $(\pm 1, 0)$ are saddle points.