Week #4

Some problems and solutions selected or adapted from Stewart Calculus.

Gradients and the Directional Derivative

In Problems 1-5 find the gradient of the given function; if a point is also given, evaluate the gradient at that specific point. Assume the variables are restricted to a domain on which the function is defined.

1. $f(x, y) = \frac{3}{2}x^5 - \frac{4}{7}y^6$

$$\begin{aligned} f_x &= \frac{15}{2}x^4 \\ f_y &= \frac{-24}{7}y^5 \\ \text{so } \nabla f &= \left\langle \frac{15}{2}x^4, \frac{-24}{7}y^5 \right\rangle \end{aligned}$$

2. $f(x, y, z) = 1/(x^2 + y^2 + z^2)$

$$\begin{aligned} f_x &= \frac{-2x}{(x^2+y^2+z^2)^2} \\ f_y &= \frac{-2y}{(x^2+y^2+z^2)^2} \\ f_z &= \frac{-2z}{(x^2+y^2+z^2)^2} \\ \text{So grad } f &= \frac{-2}{(x^2+y^2+z^2)^2}\langle x,\ y,\ z\rangle \end{aligned}$$

3. $f(x, y, z) = xe^y \sin z$

$$\begin{aligned} f_x &= e^y \sin z \\ f_y &= xe^y \sin z \\ f_z &= xe^y \cos z \\ \text{So grad } f &= \langle e^y \sin z,\ xe^y \sin z,\ xe^y \cos z\rangle \end{aligned}$$

4. $f(x, y) = \sqrt{x^2 + y^2}$

$f(x, y) = (x^2 + y^2)^{1/2}$

$$\begin{aligned} f_x &= \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} \\ \text{similarly,} \quad f_y &= \frac{y}{\sqrt{x^2+y^2}} \\ \text{so } \nabla f &= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \end{aligned}$$

5. $z = \sin(x/y)$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \cos\left(\frac{x}{y}\right)\frac{1}{y} \\ \frac{\partial z}{\partial y} &= \cos\left(\frac{x}{y}\right)\frac{-x}{y^2} \\ \text{so } \nabla f &= \left\langle \frac{1}{y}\cos\left(\frac{x}{y}\right), \frac{-x}{y^2}\cos\left(\frac{x}{y}\right) \right\rangle \end{aligned}$$

In Problems 6-7, find the directional derivative $f_{\vec{u}}(1, 2)$ for the function f with $\vec{u} = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$.

6. $f(x, y) = 3x - 4y$

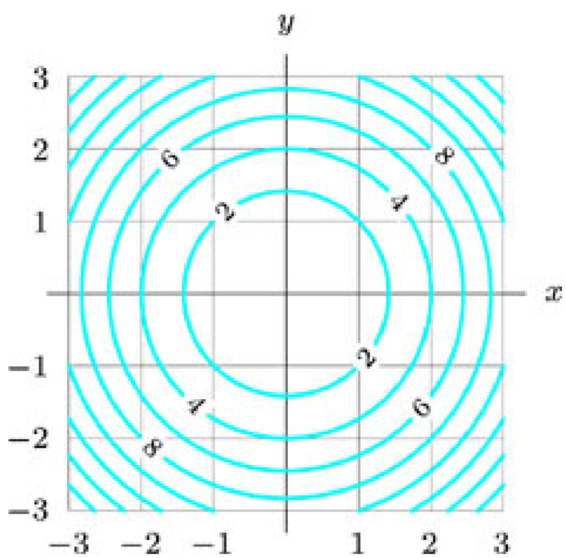
We note that the vector $\vec{u}$ is already a unit vector, since $||\vec{u}|| = \sqrt{\frac{3^2}{5^2} + \frac{(-4)^2}{5^2}} = \sqrt{\frac{25}{25}} = 1$. This means that we can compute the directional derivative using the simple formula $f_{\vec{u}}(1, 2) = [\nabla f(1, 2)] \cdot \vec{u}$.

$$\begin{aligned} f_x &= 3 \\ f_y &= -4 \\ \text{so} \quad \nabla f(1,2) &= \langle 3, -4\rangle \\ \text{So} \quad f_{\vec{u}}(1,2) &= (\nabla f(1,2)) \cdot \vec{u} \\ &= \langle 3, -4\rangle \cdot \left\langle \frac{1}{5}(3, -4) \right\rangle \\ &= \frac{1}{5}(9 + 16) \\ &= 5 \end{aligned}$$

7. $f(x, y) = xy + y^3$

$$
\begin{aligned}
f_x &= y \\
f_y &= x + 3y^2 \\
\text{so} \quad \nabla f(1,2) &= \langle 2, 1 + 3(2^2) \rangle = \langle 2, 13 \rangle \\
\text{So} \quad f_{\vec{u}}(1,2) &= \langle 2, 13 \rangle \cdot \left\langle \frac{1}{5}(3, -4) \right\rangle \\
&= \frac{1}{5}(6 - 52) = \frac{-46}{5}
\end{aligned}
$$

In Problems 8-13, use the contour diagram of $f(x, y)$ shown below to decide if the specified directional derivative is positive, negative, or approximately zero.



8. At the point $(-2, 2)$, in direction $\langle 1, 0 \rangle$.

Due east/right on the map. Moving from contour $z = 8$ towards contour $z = 6$ means z is decreasing in that direction, so the directional derivative is negative.

9. At the point $(0, -2)$, in direction $\langle 0, 1 \rangle$.

Due north/up on the map. Moving from $z = 4$ towards $z = 2$, so directional derivative is negative.

10. At the point $(-1, 1)$, in direction $\langle 1, 1 \rangle$.

Remember: the directional derivative gives the slope for a **very small** (infinitesimal) step in the specified direction.

If we are at the point $(-1, 1)$, the direction $\langle 1, 1 \rangle$ is parallel to the contour at that point. A very small step then keeps us on the contour, meaning our z value would be unchanging at that instant. If z does not change for a small step in that direction, then the directional derivative is zero.

11. At the point $(-1, 1)$, in direction $\langle -1, 1 \rangle$.

Moving towards higher z values, so the directional derivative is positive.

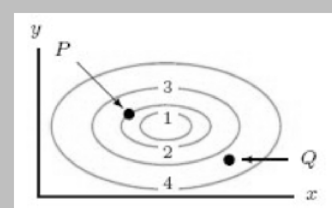
12. At the point $(0, -2)$, in direction $\langle 1, 2 \rangle$.

Moving more in the y direction than x, or towards lower z values, so the directional derivative is negative.

13. At the point $(0, -2)$, in direction $\langle 1, -2 \rangle$.

Moving towards higher z values, so the directional derivative is positive.

14. The contour diagram below represents the level curves $f(x, y)$.



In each of the following parts, decide whether the given quantity is positive, negative or zero. Explain your answer.

(a) The value of $\nabla f \cdot \langle 1, 0 \rangle$ at P.

(b) The value of $\nabla f \cdot \langle 0, 1 \rangle$ at P.

(c) $\dfrac{\partial f}{\partial x}$ at Q.

(d) $\dfrac{\partial f}{\partial y}$ at Q.

Note that this question is more about knowing the definition of directional derivative than about any particular calculation.

(a) $\nabla f \cdot \langle 1, 0 \rangle$ **is** the x-direction partial derivative of f:

$$
\begin{aligned}
\nabla f \cdot \langle 1, 0 \rangle &= D_{\langle 1,0 \rangle} f \\
&= \text{slope of surface in direction of } \langle 1, 0 \rangle \\
&= \text{slope of surface in pos } x \text{ direction} \\
&= f_x
\end{aligned}
$$

For a small step in the positive x direction from P, the function the function **decreases** so

$$f_x = \nabla f \cdot \langle 1, 0 \rangle < 0$$

(b) Similarly, $\nabla f \cdot \langle 0, 1\rangle$ **is** the y-direction partial derivative of f. For a small step in the positive y direction from P, the function the function **increases** so

$$f_y = \nabla f \cdot \langle 0, 1\rangle > 0$$

(c) $\dfrac{\partial f}{\partial x}$ at Q is positive because a small step in the positive x direction leads to an **increase** in f.

(d) $\dfrac{\partial f}{\partial y}$ at Q is negative because a small step in the positive y direction leads to a **decrease** in f.

15. The temperature at any point in the plane is given by the function

$$T(x, y) = \frac{100}{x^2 + y^2 + 1}$$

where the temperature is in °C, and the positions are in metres.

(a) What shape are the level curves of T?

(b) Where on the plane is it hottest? What is the temperature at that point?

(c) Find the direction of the greatest increase in temperature at the point (3, 2). What is the magnitude of that greatest increase (including units)?

(d) Find the direction of the greatest decrease in temperature at the point (3, 2).

(e) Find a direction at the point (3, 2) in which the temperature does not increase or decrease.

(a) The level curves of T are circles, since if T is a constant values, say c:

$$\begin{aligned} c &= \frac{100}{x^2 + y^2 + 1} \\ x^2 + y^2 + 1 &= \frac{100}{c} \\ \text{so} \quad x^2 + y^2 &= \underbrace{\frac{100}{c} - 1}_{r^2} \end{aligned}$$

which is the formula for a circle in 2D.

(b) The plane will be hottest when the denominator is smallest. From the form of the denominator, this will be when x^2 and y^2 are smallest, or at $(x, y) = (0, 0)$. Any other point (x, y) leads to a smaller temperature.

(c) To find the direction of maximum temperature increase at (3, 2), we need the gradient vector.

$$\begin{aligned} T_x &= \frac{-200x}{(x^2 + y^2 + 1)^2} \\ T_y &= \frac{-200y}{(x^2 + y^2 + 1)^2} \\ \nabla T(3, 2) &= \left\langle \frac{-600}{196}, \frac{-400}{196} \right\rangle \end{aligned}$$

or, to see the direction more easily,

$$\nabla T(3, 2) = \frac{200}{196} \langle -3, -2\rangle = \frac{50}{49} \langle -3, -2\rangle$$

The steepness of the slope in this direction is given by $||\nabla T||$:

$$||\nabla T(3, 2)|| = \frac{50}{49} || \langle -3, -2\rangle || = \frac{50}{49}\sqrt{13} \approx 3.68$$

At (3, 2), if you move towards the origin (in the direction of $\langle -3, -2\rangle$), the temperature will increase at a rate of 3.68 °C/m.

(d) If the direction of maximum temperature increase is $\langle -3, -2\rangle$, then going in the opposite direction, $\langle 3, 2\rangle$, will produce the most rapid temperature *decrease.*

(e) The direction in which the temperature does not change is perpendicular to gradient, i.e. in either the direction $\langle 2, -3\rangle$ or $\langle -2, 3\rangle$.