Week #3

Some problems and solutions selected or adapted from Stewart Calculus.

Tangent Lines and Intersections

1. Find an equation of the tangent plane to the given surface at the specified point.

   (a) $z = 3y^2 - 2x^2 + x$, $(2, -1, -3)$

   (b) $z = \sqrt{xy}$, $(1, 1, 1)$

(a) $z = f(x, y) = 3y^2 - 2x^2 + x \Rightarrow f_x(x, y) = -4x + 1$, $f_y(x, y) = 6y$, so $f_x(2, -1) = -6$.
If f has continuous partial derivatives, an equaton of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus an equation of the tangent plane is

$$\begin{aligned} &z - (-3) = f_x(2, -1)(x - 2) + f_y(2, -1)[y - (-1)] \\ \Rightarrow &z + 3 = -7(x - 2) - 6(y + 1) \text{ or} \\ &z = -7x - 6y + 5. \end{aligned}$$

(b) $z = f(x, y) = \sqrt{xy} \Rightarrow$
$f_x(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{1}{2}\sqrt{y/x}$,
$f_y(x, y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{1}{2}\sqrt{x/y}$, so $f_x(1, 1) = \frac{1}{2}$ and $f_y(1, 1) = \frac{1}{2}$. Thus an equation of the tangent plane is $z - 1 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \Rightarrow$ $z - 1 = \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1)$ or $x + y - 2z = 0$.

2. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

$f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$$f_x(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

$$f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \text{ and}$$

$$f_z(x, y, z) = \frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

so $f_x(3, 2, 6) = \frac{3}{7}$, $f_y(3, 2, 6) = \frac{2}{7}$, $f_z(3, 2, 6) = \frac{6}{7}$. Then the linear approximation of f at $(3, 2, 6)$ is given by

$$\begin{aligned} f(x, y, z) \approx L(x, y, z) &= f(3, 2, 6) + f_x(3, 2, 6)(x - 3) \\ &\quad + f_y(3, 2, 6)(y - 2) + f_z(3, 2, 6)(z - 6) \\ &= 7 + \frac{3}{7}(x - 3) + \frac{2}{7}(y - 2) + \frac{6}{7}(z - 6). \end{aligned}$$

Using this approximation, we get result

$$\begin{aligned} &\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \\ &\approx L(3.02, 1.97, 5.99) \\ &= 7 + \frac{3}{7}(3.02 - 3) + \frac{2}{7}(1.97 - 2) \\ &\quad + \frac{6}{7}(5.99 - 6) \\ &\approx 6.9914. \end{aligned}$$

Note that this approximation is simple and could even be calculated by hand.

We can compare that approximation to the exact value of $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$, noting that we definitely need a calculator for this value, reported to 5 sig figs:

$$\begin{aligned} &\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} \\ &= 6.9915 \end{aligned}$$

So the much simpler approximation is accurate to almost 4 digits after the decimal.

3. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

   Note: here Δz is the actual change in z using the equation above, and dz is the change using the linearization $L(x, y)$ at (1,2).

$dx = \Delta x = 0.05$, $dy = \Delta y = 0.01$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$,

$$\begin{aligned} dz &= z_x(1, 2)dx + z_y(1, 2)dy \\ &= (10)(0.05) + (4)(0.1) = 0.9 \end{aligned}$$

while

$$\begin{aligned} \Delta z &= f(1.05, 2.1) - f(1, 2) \\ &= 5(1.05)^2 + (2.1)^2 - 5 - 4 \\ &= 0.9225. \end{aligned}$$

4. The length and width of a rectangle are measured as 30cm and 24cm, respectively, with an error in measurement of at most 0.1cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

$dA = \dfrac{\partial A}{\partial x}\ dx + \dfrac{\partial A}{\partial y}\ dy = y\ dx$ and $|\Delta x| \leq 0.1$, $|\Delta y| \leq 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\,\text{cm}^2$.

5. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$
If the resistances are measured in ohms as $R_1 = 25\,\Omega$, $R_2 = 40\,\Omega$, $R_3 = 50\,\Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:
$$\begin{aligned}
&\frac{\partial}{\partial R_1}\left(\frac{1}{R}\right) = \frac{\partial[(1/R_1) + (1/R_2) + (1/R_3)]}{\partial R_1} \\
&\Rightarrow -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \\
&\Rightarrow \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.
\end{aligned}$$
Then by symmetry, the other partial derivatives will have the same form:
$\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, and $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$.

When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200}$ $\Leftrightarrow R = \frac{200}{17}\Omega$. Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So
$$\begin{aligned}
\Delta R &\approx dR \\
&= \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 + \frac{\partial R}{\partial R_3}\Delta R_3 \\
&= \frac{R^2}{R_1^2}\Delta R_1 + \frac{R^2}{R_2^2}\Delta R_2 + \frac{R^2}{R_3^2}\Delta R_3 \\
&= R^2\left[\frac{1}{R_1^2}\Delta R_1 + \frac{1}{R_2^2}\Delta R_2 + \frac{1}{R_3^2}\Delta R_3\right]
\end{aligned}$$
Recalling that each of the delta/error terms are 0.5% of the original resistance, or $(0.005)R_i$, e.g. $\Delta R_1 = (0.005 \cdot 25)$,
$$\begin{aligned}
\Delta R &\approx dR \\
&= \left(\frac{200}{17}\right)^2\left[\frac{1}{25^2}\underbrace{(0.005\cdot 25)}_{\Delta R_1} + \frac{1}{40^2}\underbrace{(0.005\cdot 40)}_{\Delta R_2} + \frac{1}{50^2}\underbrace{(0.005\cdot 50)}_{\Delta R_3}\right] \\
&= \left(\frac{200}{17}\right)^2(0.005)\left[\frac{1}{25} + \frac{1}{40} + \frac{1}{50}\right] \\
&\approx 0.0588
\end{aligned}$$

The Chain Rule

For questions 6-7 use the Chain Rule to find the indicated partial derivatives.

6. $z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = stu^2;$
$\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial u}$ when $s = 4, t = 2, u = 1$

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= (4x^3 + 2xy)(1) + (x^2)(tu^2) \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= (4x^3 + 2xy)(2) + (x^2(su^2), \\
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= (4x^3 + 2xy)(-1) + (x^2)(2stu)
\end{aligned}$$

When $s = 4$, $t = 2$, and $u = 1$ we have $x = 7$ and $y = 8$, so
$$\begin{aligned}
\frac{\partial z}{\partial s} &= (1484)(1) + (49)(2) = 1582, \\
\frac{\partial z}{\partial t} &= (1484)(2) + (49)(4) = 3164, \\
\frac{\partial z}{\partial u} &= (1484)(-1) + (49)(16) = -700
\end{aligned}$$

7. $w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$
$\dfrac{\partial w}{r}, \dfrac{\partial w}{\theta}$ when $r = 2, \theta = \pi/2$

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (y+z)(\cos\theta) + (x+z)(\sin\theta) + (y+x)(\theta) \\
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta} \\
&= (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)(r).
\end{aligned}$$

When $r = 2$ and $\theta = \pi/2$ we have $x = 0$, $y = 2$, and $z = \pi$ so

$$\frac{\partial w}{\partial r} = (2+\pi)(0) + (0+\pi)(1) + (2+0)(\pi/2)$$
$$= 2\pi$$
$$\frac{\partial w}{\partial \theta} = (2+\pi)(-2) + (0+\pi)(0) + (2+0)(2)$$
$$= -2\pi$$

8. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2,3) = 4$ and $T_y(2,3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$. After 3 seconds,

$$x = \sqrt{1+t} = \sqrt{1+3} = 2,$$
$$y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3,$$
$$\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}} = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}, \text{ and}$$
$$\frac{dy}{dt} = \frac{1}{3}$$

Then,

$$\frac{dT}{dt} = T_x(2,3)\frac{dx}{dt} + T_y(2,3)\frac{dy}{dt}$$
$$= 4(\frac{1}{4}) + 3(\frac{1}{3})$$
$$= 2$$

Thus the temperature is rising at a rate of 2°C/s.

9. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $PV = (8.31)T$ to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

$$\frac{dP}{dt} = 0.05,\ \frac{dT}{dt} = 0.15,\ V = 8.31\frac{T}{P},\ \text{and}$$

$$\frac{dV}{dt} = \frac{8.31}{P}\frac{dT}{dt} - 8.31\frac{T}{P^2}\frac{dP}{dt}.$$

Thus, when $P = 20$ and $T = 320$,

$$\frac{dV}{dt} = 8.31\left[\frac{0.15}{20} - \frac{(0.05)(320)}{400}\right] \approx -0.27 \text{ L/s}.$$