Week #2

Some problems and solutions selected or adapted from Stewart Calculus.

Partial Derivatives

For questions 1-5, find the first partial derivatives of the function.

1. $f(x,t) = e^{-t}\cos \pi x$

$$\begin{aligned} f_x(x,t) &= e^{-t}(-\sin \pi x)(\pi) \\ &= -\pi e^{-t} \sin \pi x \\ f_t(x,t) &= e^{-t}(-1)\cos \pi x \\ &= -e^{-t}\cos \pi x \end{aligned}$$

2. $z = (2x+3y)^{10}$

$$\begin{aligned} \frac{\partial z}{\partial x} &= 10(2x+3y)^9 \\ &= 20(2x+3y)^9 \\ \frac{\partial z}{\partial y} &= 10(2x+3y)^9 \cdot 3 \\ &= 30(2x+3y)^9 \end{aligned}$$

3. $f(x,y) = \dfrac{x}{y}$

$$\begin{aligned} & f(x,y) = xy^{-1} \\ \Rightarrow \quad & f_x(x,y) = y^{-1} = \frac{1}{y}, \\ & f_y(x,y) = -xy^{-2} = \frac{-x}{y^2} \end{aligned}$$

4. $f(x,y,z) = xz - 5x^2y^3z^4$

$$\begin{aligned} f_x(x,y,z) &= z - 10xy^3z^4, \\ f_y(x,y,z) &= -15x^2y^2z^4, \\ f_z(x,y,z) &= x - 20x^2y^3z^3 \end{aligned}$$

5. $w = \ln(x+2y+3z)$

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{x+2y+3z}, \\ \frac{\partial w}{\partial y} &= \frac{2}{x+2y+3z}, \\ \frac{\partial w}{\partial z} &= \frac{3}{x+2y+3z} \end{aligned}$$

For questions 6-7, use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$.

6. $x^2 + 2y^2 + 3z^2 = 1$

$$\begin{aligned} & \frac{\partial}{\partial x}(x^2+2y^2+3z^2) = \frac{\partial}{\partial x}(1) \\ \Rightarrow \quad & 2x + 0 + 6z\frac{\partial z}{\partial x} = 0 \\ \Rightarrow \quad & 6z\frac{\partial z}{\partial x} = -2x \\ \Rightarrow \quad & \frac{\partial z}{\partial x} = \frac{-2x}{6z} = -\frac{x}{3z}, \quad \text{and} \\ & \frac{\partial}{\partial y}(x^2+2y^2+3z^2) = \frac{\partial}{\partial y}(1) \\ \Rightarrow \quad & 0 + 4y + 6z\frac{\partial z}{\partial y} = 0 \\ \Rightarrow \quad & 6z\frac{\partial z}{\partial y} = -4y \\ \Rightarrow \quad & \frac{\partial z}{\partial y} = \frac{-4y}{6z} = -\frac{2y}{3z} \end{aligned}$$

7. $e^z = xyz$

$$\begin{aligned} & \frac{\partial}{\partial x}(e^z) = \frac{\partial}{\partial x}(xyz) \\ \Rightarrow \quad & e^z\frac{\partial z}{\partial x} = y\left(x\frac{\partial z}{\partial x} + z\cdot 1\right) \\ \Rightarrow \quad & e^z\frac{\partial z}{\partial x} - xy\frac{\partial z}{\partial x} = yz \\ \Rightarrow \quad & (e^z - xy)\frac{\partial z}{\partial x} = yz \\ \Rightarrow \quad & \frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}, \quad \text{and} \\ & \frac{\partial}{\partial y}(e^z) = \frac{\partial}{\partial y}(xyz) \\ \Rightarrow & e^z\frac{\partial z}{\partial y} = x\left(y\frac{\partial z}{\partial y} + z\cdot 1\right) \\ \Rightarrow & e^z\frac{\partial z}{\partial y} - xy\frac{\partial z}{\partial y} = xz \\ \Rightarrow & (e^z = xy)\frac{\partial z}{\partial y} = xz \\ \Rightarrow & \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}. \end{aligned}$$

For question 8, find all the second partial derivatives of the given function.

8. $z = \arctan\left(\dfrac{x+y}{1-xy}\right)$

Recall that the derivative rule for arctan(something) is $\frac{1}{1+(\text{the thing})^2}$ followed by the chain rule, multiplying by derivative of the inside. With this function, that 'inside' derivative requires using the quotient rule.

$$z_x = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)-(x+y)(-y)}{(1-xy)^2}$$

This is fine if we just need an expression for the first derivative, but since we need to differentiate again, we should simplify if we can (and we can simplify it a lot as it turns out!)

$$\begin{aligned}
\text{tidying, } z_x &= \frac{1}{\left(\frac{1-xy}{1-xy}\right)^2+\left(\frac{x+y}{1-xy}\right)^2}\left(\frac{1+y^2}{(1-xy)^2}\right)\\
&= \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2}\left(\frac{1+y^2}{(1-xy)^2}\right)\\
&= \frac{1+y^2}{(1-xy)^2+(x+y)^2}\\
&= \frac{1+y^2}{1+x^2+y^2+x^2y^2}\\
\text{factoring, } &= \frac{1+y^2}{(1+x^2)(1+y^2)}\\
&= \frac{1}{1+x^2}.
\end{aligned}$$

Contour Diagrams and Partial Derivatives

Following a similar sequence, we can find the y partial derivative, then simplify it as well.

$$\begin{aligned}
z_y &= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1)(1-xy)-(x+y)(-x)}{(1-xy)^2}\\
&= \frac{1+x^2}{(1-xy)^2+(x+y)^2}\\
&= \frac{1+x^2}{(1+x^2)(1+y^2)}\\
&= \frac{1}{1+y^2}.
\end{aligned}$$

With the relatively simple expressions for the first partial derivatives, we can now differentiate again and get the second partial derivatives.

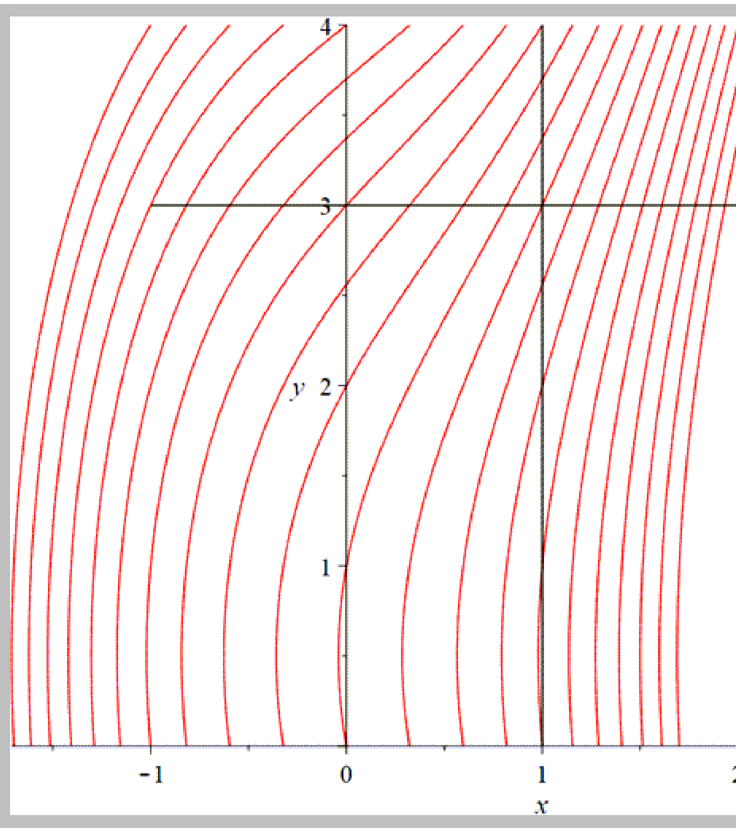
$$\begin{aligned}
z_{xx} &= -(1+x^2)^{-2}\cdot 2x\\
&= -\frac{2x}{(1+x^2)^2},\\
z_{xy} &= z_{yx} = 0,\ \text{and}\\
z_{yy} &= -(1+y^2)^{-2}\cdot 2y\\
&= -\frac{2y}{(1+y^2)^2}
\end{aligned}$$

To the right is the contour diagram for a surface $z = f(x, y)$. The contour lines are spaced at intervals of $\Delta z = 1$, and they increase as we move to the right. Also $f(0, 0) = 0$.

(a) Use the diagram to estimate the partial derivatives $f_x(1, 3)$ and $f_y(1, 3)$. For reference, the point (1,3) is at the cross-hairs on the contour diagram.

(b) Use the diagram to find the sign of the second partial derivatives

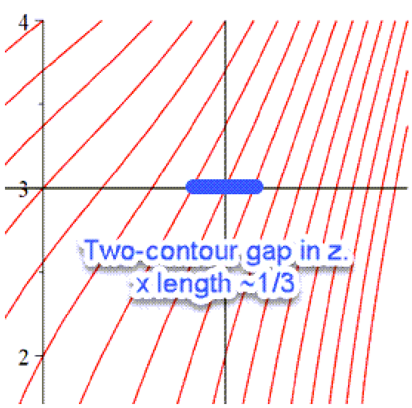
- $f_{xx}(1, 3)$,
- $f_{yy}(1, 3)$, and
- $f_{xy}(1, 3)$.

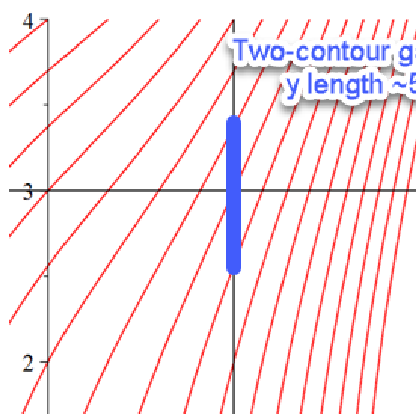


(a) Use the contour diagram to estimate the partial derivatives, using a rise-over-run ratio. Note that the 'run' is the distance on the xy plane, shown in the contour diagram. The 'rise' comes from the change in f or z, which is indicated by counting how many contour lines are between two points, and whether the net height change is positive or negative.

To keep the steps small (like a derivative), we usually measure a rise-over-run from our start point, here (1,3), out to the nearest contours. For the x partial derivative estimate, we 'run' in the x direction, and for the y partial derivative estimate we 'run' in the y direction.

Here, we took a **step to either side of our starting point** to the next contour, to increase our length measurement accuracy; another reasonable estimate though would be measuring between the starting point and the next contour just in one direction.





Based on the contour diagram measurements shown in the diagrams above, this gives us:

- $f_x(1, 3) = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta z}{\Delta x} \approx \dfrac{2}{1/3} = 6$
- $f_y(1, 3) = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta z}{\Delta y} \approx \dfrac{-2}{5/6} = -2.4$

(b)
- $f_{xx}(1, 3)$ is a measure of the rate of change of f_x as we increase x (leaving y fixed). As we move to the right from the point (1, 3) the contour lines in the x-direction get closer together and that means that the slope in the

x-direction gets steeper and so f_x increases.
Thus $f_{xx}(1,3) > 0$.

- $f_{yy}(1,3)$. Similarly f_{yy} is a measure of the rate of change of f_y, as we increase y (leaving x fixed). As we move upwards from the point (1, 3) the contour lines in the y-direction) get closer together and that means that the slope in the y-direction gets steeper and so as a result f_y (which is negative!) decreases (gets more negative).
Thus $f_{yy}(1,3) < 0$
- $f_{xy}(1,3)$ is a measure of the rate of change of f_x as we increase y (leaving x fixed). As we move upwards from the point (1, 3) the spacing in the contour lines in the x-direction) does not really seem to change. That means that the slope in the x-direction doesn't change.
Thus $f_{xy}(1,3) = 0$.