APSC 172 Practice Problems

Week #1

Some problems and solutions selected or adapted from Stewart Calculus.

Equations of Lines and Planes

1. Find a parametric equation for the line through the point $(1, 0, 6)$ and perpendicular to the plane $x + 3y + z = 5$.

A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1\rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Then a vector equation is

$$\begin{aligned}\mathbf{r} &= (\mathbf{i} + 6\mathbf{k}) + t(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= (1 + t)\mathbf{i} + (0 + 3t)\mathbf{j} + (6 + t)\mathbf{k}\end{aligned}$$

and parametric equations are $x = 1 + t$, $y = 3t$, $z = 6 + t$.

2. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$L_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$$

$$L_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

Since the direction vectors $\langle 1, -2, -3\rangle$ and $\langle 1, 3, -7\rangle$ aren't scalar multiples of each other, the lines aren't parallel. Parametric equations of the lines are $L_1 : x = 2 + t, y = 3 - 2t, z = 1 - 3t$ and $L_2 : x = 3 + s, y = -4 + 3s, z = 2 - 7s$. Thus, for the lines to intersect, the three equations $2 + t = 3 + s$, $3 - 2t = -4 + 3s$, and $1 - 3t = 2 - 7s$ must be satisfied simultaneously. Solving the first two equations gives $t = 2$, $s = 1$ and checking, we see that these values do satisfy the third equation, so the lines intersect when $t = 2$, $s = 1$, that is, at the point $(4, -1, -5)$.

3. For each part below, find an equation of the plane matching the given description.

   (*a*) The plane through the point $(-1, \frac{1}{2}, 3)$ and with normal vector $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

   (*b*) The plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$

   (*c*) The plane through the point $(1, \frac{1}{2}, \frac{1}{3})$ and parallel to the plane $x + y + z = 0$

   (*d*) The plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$

   (*e*) The plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$

(*a*) $\mathbf{i}+4\mathbf{j}+\mathbf{k} = \langle 1, 4, 1\rangle$ is a normal vector to the plane and $(-1, \frac{1}{2}, 3)$ is a point on the plane, so setting $a = 1$, $b = 4$, $c = 1$, $x_0 = -1$, $y_0 = \frac{1}{2}$, $z_0 = 3$ in the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

gives

$$1[x - (-1)] + 4\left(y - \frac{1}{2}\right) + 1(z - 3) = 0$$

or $x + 4y + z = 4$ as an equation of the plane.

(*b*) Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 5, -1, -1\rangle$, and an equation of the plane is $5(x - 1) - 1[y - (-1)] - 1[z - (-1)] = 0$ or $5x - y - z = 7$.

(*c*) Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 1, 1, 1\rangle$, and an equation of the plane is $1(x-1)+1(y-\frac{1}{2})+1(z-\frac{1}{3}) = 0$ or $x+y+z = \frac{11}{6}$.

(*d*) First we find two nonparallel vectors in the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4\rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6-4, 0-3, -2-7\rangle = \langle 2, -3, -9\rangle$. Using dot product or cross products, a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{n} = \langle -33, -10, -4\rangle.$$

Thus, an equation of the plane is $-33(x-6)-10(y-0)-4[z-(-2)] = 0$ or, rearranging and gathering terms, $33x+10y+4z-190 = 0$.

(*e*) A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1\rangle \times \langle 2, -1, 3\rangle = \langle 2, -5, -3\rangle$, and $\mathbf{a}$ is parallel to the desired plane.

Another vector parallel to the plane, call it $\vec{b}$, is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. To find a single point on the line, we arbitrarity set $x = 0$, and sub that into the equations of the planes to find one of the intersection points which lines on the intersection line:
$x = 0:\quad y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $(0, \frac{7}{2}, \frac{3}{2})$.
A vector $\vec{b}$ connecting our known points on the plane then is

$$\begin{aligned}\vec{b} &= \left[0, \frac{7}{2}, \frac{3}{2}\right] - [-1, 2, 1] \\ &= \left[1, \frac{3}{2}, \frac{1}{2}\right]\end{aligned}$$

For the plane we want then, we now have:

- A reference point $(-1, 2, 1)$, and
- two vectors parallel to the plane, $\vec{a} = [2, -5, -3]$ and $\vec{b} = \left[1, \frac{3}{2}, \frac{1}{2}\right]$.

That information defines our plane conceptually, but isn't in a format that would let us build a formula for the plane. To get a formula, it's simpler if we have a point and a *normal* vector; then we can use the normal-vector- components-to-plane-coefficients construction.

Fortunately, knowing two vectors *in* the plane, $\vec{a}$ and $\vec{b}$, we can build a vector *normal* to the plane with another cross-product calculation:

$$\begin{aligned}\vec{n} = \vec{a} \times \vec{b} &= [2, -5, -3] \times \left[1, \frac{3}{2}, \frac{1}{2}\right] \\ &= [2, -4, 8]\end{aligned}$$

Now with the reference point $(-1, 2, 1)$ and the normal vector $\vec{n} = [2, -4, 8]$ for our desired plane, we can finally build the plane's equation:

$$\begin{aligned}2(x+1) - 4(y-2) + 8(z-1) = 0 \text{ or} \\ 2x - 4y + 8z = -2 \text{ or dividing by 2,} \\ x - 2y + 4z = -1\end{aligned}$$

Slopes of Trajectories

4. We use the in-class definition of the slope of a 3D parametric curve as the z rise over the xy run of the velocity vector: $m = \dfrac{\dot{z}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}}$.

   Consider a particle whose projected path on the xy plane follows $[x, y] = [2t, 3t-1]$, as it passes over the surface defined by $z = x - y^2$.

   (a) Find the xyz location of the particle at $t = 2$, and then determine if the particle is moving uphill or downhill.

   (b) Find the xyz location of the particle at $t = -1$, and find the instantaneous slope at that time.

(a) At $t = 2$, $x = 2t = 4$, $y = 3t - 1 = 5$, and $z = x - y^2 = 4 - 5^2 = -21$.
The full coordinate location is $[x, y, z] = [4, 5, -21]$.

To find whether the particle is moving uphill or downhill at that moment, we need to compute $\dot{z} = \frac{dz}{dt}$.

We can find this either through the chain rule, or substitution to build $z(t)$.

- Chain rule: $z = x - y^2$, so $\frac{d}{dt}z = \frac{dx}{dt} - 2y\frac{dy}{dt}$.
  Since $x = 2t$, then $\frac{dx}{dt} = 2$. And since $y = 3t - 1$, $\frac{dy}{dt} = 3$:
  $\frac{dz}{dt} = 2 - 2y(3)$.
  At $t = 2$, we found $y = 5$, so
  $\frac{dz}{dt} = 2 - 2(5)(3) = -28$.
- Substitution: $z = x - y^2$, but $x = 2t$ and $y = 3t - 1$, so
  $z = 2t - (3t-1)^2$.
  Differentiating with respect to time,
  $\frac{dz}{dt} = 2 - 2(3t-1)(3)$.
  At $t = 2$,
  $\frac{dz}{dt} = 2 - 2(3(2) - 1)(3) =$ -28.

Since the rate of change $\frac{dz}{dt}$ is negative, the particle is moving downhill.

(b) At $t = -1$, $x = 2t = -2$, $y = 3t - 1 = -4$, and $z = x - y^2 = -2 - (-4)^2 = -2 - 16 = -18$.
The full coordinate location is $[x, y, z] = [-2, -4, -18]$.

To find the slope, we need all three time derivatives, $\dot{x}$, $\dot{y}$ and $\dot{z}$.

$$\begin{aligned}\dot{x} &= \frac{d}{dt}(2t) = 2 \\ \dot{y} &= \frac{d}{dt}(3t-1) = 3 \\ \dot{z} &= \frac{d}{dt}x - y^2\end{aligned}$$

Both $\dot{x} = 2$ and $\dot{y} = 3$ are constant, so don't depend on time $t = -1$.
The final derivative $\dot{z}$ can, as in part (a), be computed using either the chain rule or substitutions. We'll show the chain rule applied at $t = -1$, starting with $z = x - y^2$:

$$\begin{aligned} \frac{d}{dt}z &= \frac{dx}{dt} - 2y\frac{dy}{dt} \\ &= 2 - 2y(3). \\ \text{at } t = -1, y = -4: \ &= 2 - 2(-4)(3) = 26 \end{aligned}$$

Putting all the derivatives together, we find the slope

$$\begin{aligned} m &= \frac{\dot{z}}{\sqrt{(\dot{x})^2 + (\dot{y})^2}} \\ &= \frac{26}{\sqrt{2^2 + 3^2}} \\ &= \frac{26}{\sqrt{13}} \approx 7.21 \end{aligned}$$

The particle is moving uphill with a slope of approximately 7.21.