Week #11 - Differential Equations and Complex Numbers - Part II

Some problems and solutions selected or adapted from Stewart Calculus, or Zill and Shanahan Complex Analysis.

The Complex Exponential Function

1. Write the following numbers in the form $re^{i\theta}$, using an angle between 0 and 2π.
   (a) $-3+3i$
   (b) $1-\sqrt{3}i$
   (c) $3+4i$
   (d) $8i$
2. Write the following complex exponentials in $z = a + bi$ form.
   (a) $e^{i\pi/2}$
   (b) $e^{i2\pi}$
   (c) $e^{i\pi/3}$
   (d) $e^{-i\pi}$
   (e) $2e^{i\pi}$
   (f) $4e^{i}$
   (g) $10e^{0.6435\ i}$
   (h) $e^{(3-2i)t}$
   (i) $e^{(-2-3i)t}$

Spring/Mass Systems

3. (a) Use the assumption that $y(t) = e^{\lambda t}$ to find $y(t)$ if
   $$5y'' + 33y = 0,$$
   given the initial conditions
   $y(0) = 9$ and $y'(0) = 4$.
   (b) Interpreting the equation as the model for a spring/mass system, describe both characteristics of the system (mass, spring constant and damping coefficient), and the type of motion it will exhibit (undamped, underdamped, critically damped or overdamped).
4. (a) Use the assumption that $s(t) = e^{\lambda t}$ to find $s(t)$ if
   $$s'' + 2s' + 2s = 0,$$
   given the initial conditions
   $s(0) = 0$ and $s'(0) = 5$.
   (b) Interpreting the equation as the model for a spring/mass system, describe both characteristics of the system (mass, spring constant and damping coefficient), and the type of motion it will exhibit (undamped, underdamped, critically damped or overdamped).
5. (a) Use the assumption that $x(t) = e^{\lambda t}$ to find $x(t)$ if
   $$1600x'' + 81x = 0,$$
   given the initial conditions
   $x(0) = 5$ and $x'(0) = 3$.
   (b) Interpreting the equation as the model for a spring/mass system, describe both characteristics of the system (mass, spring constant and damping coefficient), and the type of motion it will exhibit (undamped, underdamped, critically damped or overdamped).
6. (a) Use the assumption that $y(t) = e^{\lambda t}$ to find $y(t)$ if
   $$9y'' + 18y' + 21y = 0,$$
   given the initial conditions
   $y(0) = 4$ and $y'(0) = 5$.
   (b) Interpreting the equation as the model for a spring/mass system, describe both characteristics of the system (mass, spring constant and damping coefficient), and the type of motion it will exhibit (undamped, underdamped, critically damped or overdamped).
7. (a) Use the assumption that $x(t) = e^{\lambda t}$ to find $x(t)$ if
   $$50x'' + 20x' + x = 0,$$
   given the initial conditions
   $x(0) = 2$ and $x'(0) = 3$.
   (b) Interpreting the equation as the model for a spring/mass system, describe both characteristics of the system (mass, spring constant and damping coefficient), and the type of motion it will exhibit (undamped, underdamped, critically damped or overdamped).

8. Consider the following differential equations representing the motion of a spring/mass system. Assume all values are in the usual SI units for force, mass, position, etc.

$$\begin{aligned}
&\text{(i) } x'' + 9x = 0, && x(0) = 7, && x'(0) = 0\\
&\text{(ii) } 25x'' + x = 0, && x(0) = 28, && x'(0) = 0\\
&\text{(iii) } x'' + 25x = 0, && x(0) = 15, && x'(0) = 0\\
&\text{(iv) } 9x'' + x = 0, && x(0) = 6, && x'(0) = 0
\end{aligned}$$

   (a) Which differential equation represents the mass oscillating with the **shortest period** (looks like the fastest oscillations)? Compute the period of the oscillations for that equation.

   (b) Which differential equation represents the mass oscillating with the **longest period** (looks like the slowest oscillations)? Compute the period of the oscillations for that equation.

   (c) Which differential equation represents the mass oscillating with the **largest amplitude**? Compute the resulting amplitude.

   (d) Which differential equation represents the mass oscillating with the **smallest amplitude**? Compute the resulting amplitude.

   (e) Which differential equation represents the mass oscillating with the **highest possible velocity**? Compute the maximum possible velocity.

9. For a spring/mass system with $m = 1$ kg, $k = 16$ N/m and a variable damping coefficient c, we arrive at the following differential equation:

$$x'' + cx' + 16x = 0.$$

   Find the values of c that make the general solution overdamped, underdamped, or critically damped.

10. A mass of 100 g stretches a spring 5 cm. There is no damping.

   (a) If the mass is set in motion from its equilibrium position with a velocity of 10 cm/s, determine the position of the mass at any time.

   (b) When does the mass first return to its equilibrium position?

11. A clock designer has a 500 g weight to use in a pendulum clock. The motion of a pendulum with mass m kg and L length in meters is governed by

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

   (a) The pendulum DE in its current form cannot be solved using the techniques covered so far in the class. What approximation could be used to linearize the equation, and how would this approximation limit our interpretation of the solutions?

   (b) Use the linearization from part (a) to obtain a new differential equation, and find its general solution.

   (c) Based on your solution, find the length of pendulum that would produce oscillations with a period of 1 cycle per second (a clock maker's favourite).

   (d) If the clock maker were to use a lighter weight, how would the length of the pendulum need to change?