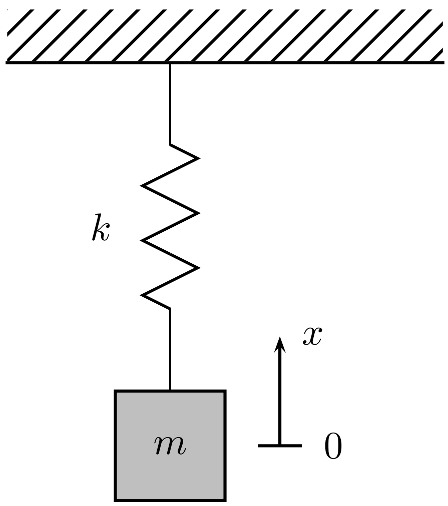
Week #10 - Differential Equations and Complex Numbers - Part 1

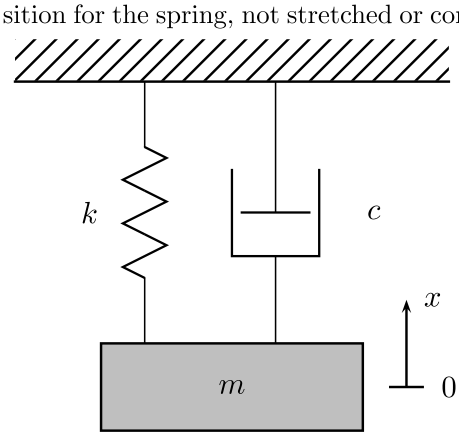
Some problems and solutions selected or adapted from Stewart Calculus, or Zill and Shanahan Complex Analysis.

Spring Systems

1. (a) Use $\sum F = ma$ to write a differential equation constraining the position of a mass over time, $x(t)$ for the spring/mass system shown below.
   Assume that $x = 0$ is the equilibrium position for the spring, not stretched or compressed.



   (b) Use $\sum F = ma$ to write a differential equation constraining the position of a mass over time, $x(t)$ for the *damped* spring/mass system shown below. Assume that the damping force is proportional to velocity.
   Further assume that $x = 0$ is the equilibrium position for the spring, not stretched or compressed.



2. The differential equations for three spring systems, (A) through (C), and four prediction for $x(t)$, (1) through (4), are given below. Match each differential equation with the correct prediction if the mass were given an initial nudge away from equilibrium. There is one graph option that won't be matched.

   Support your answer with a brief explanation.

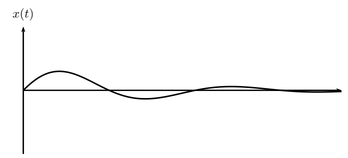
   Spring system DEs:

   (A) $x'' + 4x = 0$.

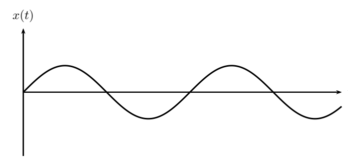
   (B) $x'' + x' + 4x = 0$.

   (C) $x'' + 8x' + 4x = 0$.

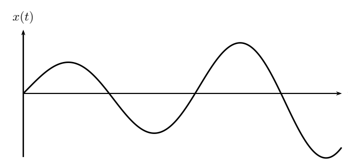
   Graphs of behaviour:



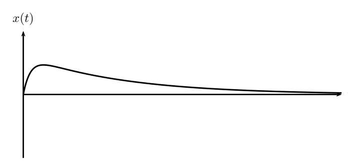
(1)



(2)



(3)



(4)

Complex Numbers

3. For each part, graph z_1 and z_2 on the complex plane, and then compute and graph the indicated sums and differences as well.

   (a) $z_1 = 4 + 2i$, $z_2 = -2 + 5i$; compute $z_1 + z_2$ and $z_1 - z_2$.

   (b) $z_1 = 1 + i$, $z_2 = 1 - i$; compute $z_1 + z_2$ and $z_1 - z_2$.

   (c) $z_1 = 4 - 3i$, $z_2 = 2$; compute $2z_1 + 4z_2$ and $z_1 - z_2$.

   (d) $z_1 = 5 + 4i$, $z_2 = -3i$; compute $3z_1 + 5z_2$ and $z_1 - 2z_2$.

4. For each part, graph z_1 and z_2, and then compute the product $z_1 \cdot z_2$ and graph it on the complex plane.

   (a) $z_1 = 2 + i$, $z_2 = i$.

   (b) $z_1 = 1 + 3i$, $z_2 = -i$.

   (c) $z_1 = -1$, $z_2 = i$.

   (d) $z_1 = 1 + i$, $z_2 = 1 - i$.

5. Write the following expression in $z = a + bi$ form.

   (a) $(2 + 5i)(4 - i)$

   (b) $(1 - 2i)(8 - 3i)$

   (c) i^3

   (d) i^{100}

(e) $\sqrt{-25}$

6. Graph the following complex numbers, and then write them in polar form ($r \angle \theta$), using an angle between 0 and 2π.

   (a) $-3+3i$

   (b) $1-\sqrt{3}i$

   (c) $3+4i$

   (d) $8i$

7. Identify which of the following pairs are complex conjugate pairs.

   (a) $z_1 = 3+2i$ and $z_2 = 3-2i$

   (b) $z_1 = -4-4i$ and $z_2 = 4+4i$

   (c) $z_1 = 5-4i$ and $z_2 = 5+4i$

   (d) $z_1 = 2$ and $z_2 = 2$

   (e) $z_1 = 2$ and $z_2 = 3$

8. Build the complex conjugate $\bar{z}$ for the following complex numbers. Report the values in the same form as the original z, i.e. the same Cartesian or polar form.

   (a) $z = 2+3i$.

   (b) $z = 4-2i$.

   (c) $z = 3 \angle \frac{\pi}{4}$.

   (d) $z = 1 \angle \frac{-\pi}{2}$.

9. Demonstrate that the following sums and products of complex conjugate pairs are all-real, with 0 imaginary component.

   (a) $(3+4i)+(3-4i)$

   (b) $(a+bi)+(a-bi)$

   (c) $(3+4i)\cdot(3-4i)$

   (d) $(a+bi)\cdot(a-bi)$