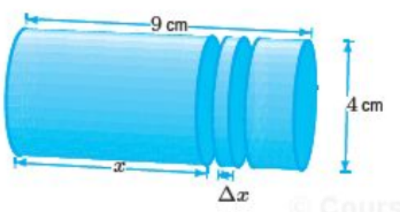
# Week #9 - Integrals for Volume and Work; Partial Fractions

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

## Volumes of Geometric Shapes

In Questions #1-6, write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly. (Regions are parts of cones, cylinders, spheres, and pyramids.)

1.

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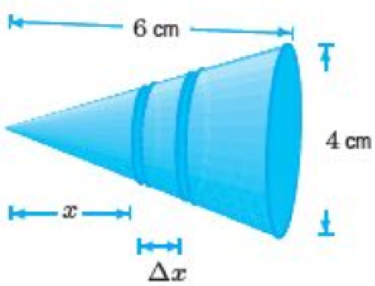


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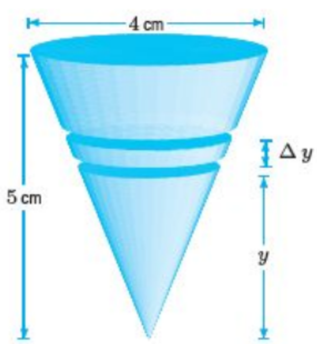
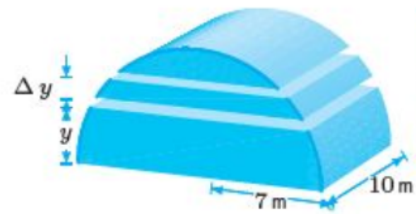



4. Note: for this example, the integral you construct cannot be evaluated using just the techniques from class. **However**, if you think of area interpretations of the integral, you can still compute the value of the integral in other ways.

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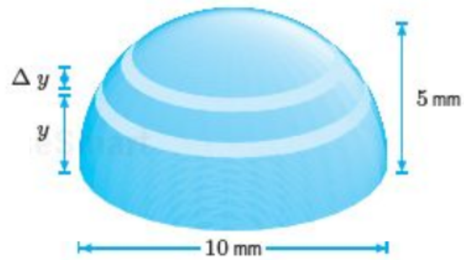


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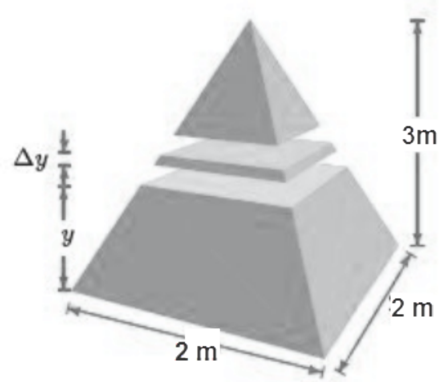
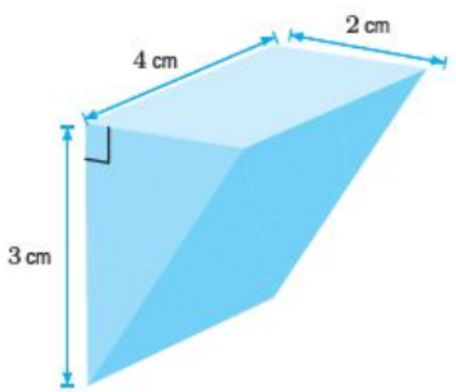



7. Find, by slicing, the volume of a cone whose height is 3 cm and whose base radius is 1 cm.

8. Find the volume of a sphere of radius $r$ by slicing.

9. Find, by slicing, a formula for the volume of a cone of height $h$ and base radius $r$.

10. The figure below shows a solid with both rectangular and triangular cross sections.




(a) Slice the solid parallel to the triangular faces. Sketch one slice and calculate its volume in terms of $x$, the distance of the slice from one end. Then write and evaluate an integral giving the volume of the solid.

(b) Repeat part (a) for horizontal slices. Instead of $x$, use $h$, the distance of a slice from the top.

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## Volumes of Revolution

In Questions #11-15, the region is rotated around the $x$-axis. Find the volume.

11. Bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 1$.

12. Bounded by $y = (x+1)^2$, $y = 0$, $x = 1$, $x = 2$.

13. Bounded by $y = 4 - x^2$, $y = 0$, $x = -2$, $x = 0$.

14. Bounded by $y = \sqrt{x+1}$, $y = 0$, $x = -1$, $x = 1$.

15. Bounded by $y = e^x$, $y = 0$, $x = -1$, $x = 1$.

16. Find the volume obtained when the region bounded by $y = x^2$, $y = 1$, and the $y$-axis is rotated around the $y$-axis.

17. Find the volume obtained when the region in the 1st (upper-right) quadrant, bounded by $y = x^2$, $y = 1$, and the $y$-axis, is rotated around the $x$-axis.

18. Find the volume obtained when the region bounded by $y = e^x$, the $x$-axis, and the lines $x = 0$ and $x = 1$ is rotated around the $x$-axis.

19. Rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the $x$-axis generates an ellipsoid. Compute its volume.

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## Partial Fractions

For problems **20-33**, evaluate the integral.

20. $\displaystyle\int \frac{x^4}{x-1}\,dx$

21. $\displaystyle\int \frac{3t-2}{t+1}\,dt$

22. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

23. $\displaystyle\int \frac{y}{(y+4)(2y-1)}\,dy$

24. $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

25. $\displaystyle\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

26. $\displaystyle\int \frac{ax}{x^2-bx}\,dx$

27. $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$. Assume that $a$ and $b$ are different constants, i.e. $a \neq b$.

28. $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

29. $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

30. $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

31. $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

32. $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$

33. $\displaystyle\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2}\,dx$

34. Evaluate the integral $\displaystyle\int x \arctan(x)\,dx$

35. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. If $P$ represents the number of female insects in a population, $S$ the number of sterile males introduced each generation, and $r$ the population's natural growth rate, then the female population is related to $t$ by

$$t = \int \frac{P+S}{P\left[(r-1)P - S\right]}\,dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for $P$.)

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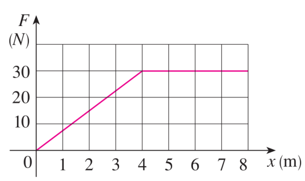
**Work**

36. When a particle is located a distance $x$ meters from the origin, a force of $F(x) = \cos\left(\frac{\pi x}{3}\right)$ Newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

37. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done by the force in moving an object a distance of 8 m?

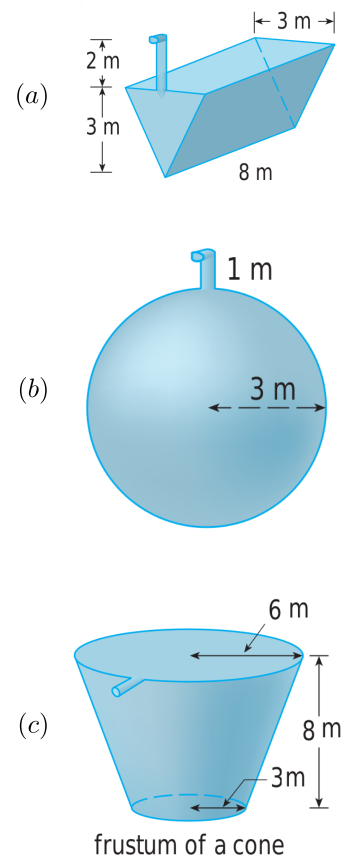



38. A force of 10 N is required to hold a spring stretched 4 cm beyond its natural length. How much work is done in stretching it from its natural length to 6 cm beyond its natural length?

39. A cable that weighs 2 kg/m is used to lift 800 kg of coal up a mine shaft 500 m deep. Find the work done.

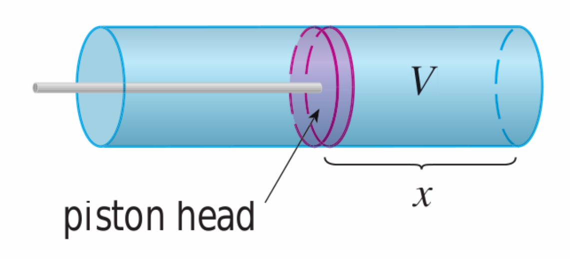
40. A leaky 10 kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs $0.8\,\mathrm{kg\,m^{-1}}$. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?

41. There are several tank designs shown below; each tank is full of water. Find the work required to pump the water out of the spout.




42. When gas expands in a cylinder with radius $r$, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area: $F = \pi r^2 P$. Show that the work done by the gas when the volume expands from volume $V_1$ to volume $V_2$ is

$$W = \int_{V_1}^{V_2} P\,dV$$


V
piston head
x