

## Week #8 - Integration by Parts and Substitution

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

### Substitution Integrals

To practice computing integrals using substitutions, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

**Note:** In the solutions to these problems, we always show the substitution used. On a test, if you can compute the antiderivative in your head, you do *not* need to go through all the steps shown here. They are included in these solutions as a learning and comprehension aid.

- $\int te^{t^2} dt$
- $\int e^{3x} dx$
- $\int e^{-x} dx$
- $\int 25e^{-0.2t} dt$
- $\int t \cos(t^2) dt$
- $\int \sin(2x) dx$
- $\int \sin(3-t) dt$
- $\int xe^{-x^2} dx$
- $\int (r+1)^3 dr$
- $\int y(y^2+5)^8 dy$
- $\int t^2(t^3-3)^{10} dt$
- $\int x^2(1+2x^3)^2 dx$
- $\int x(x^2+3)^2 dx$
- $\int x(x^2-4)^{7/2} dx$
- $\int y^2(1+y)^2 dy$
- $\int (2t-7)^{73} dt$
- $\int \frac{1}{y+5} dy$
- $\int \frac{1}{\sqrt{4-x}} dx$
- $\int (x^2+3)^2 dx$
- $\int x^2e^{x^3+1} dx$
- $\int \sin(\theta)(\cos(\theta)+5)^7 d\theta$
- $\int \sqrt{\cos(3t)} \sin(3t) dt$
- $\int \sin^6(\theta) \cos(\theta) d\theta$
- $\int \frac{1}{\sqrt{1-x^2}} dx$
- $\int \frac{1}{\sqrt{5-x^2}} dx$
- $\int \frac{x}{\sqrt{1-x^2}} dx$
- $\int \sin^3(\alpha) \cos(\alpha) d\alpha$
- $\int \sin^6(5\theta) \cos(5\theta) d\theta$
- $\int \tan(2x) dx$
- $\int \frac{1}{x^2+1} dx$
- $\int \frac{1}{x^2+20} dx$
- $\int \frac{(\ln z)^2}{z} dz$
- $\int \frac{e^t+1}{e^t+t} dt$
- $\int \frac{y}{y^2+4} dy$
- $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

36.  $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy$
37.  $\int \frac{1 + e^x}{\sqrt{x + e^x}} dx$
38.  $\int \frac{e^x}{2 + e^x} dx$
39.  $\int \frac{x + 1}{x^2 + 2x + 19} dx$
40.  $\int \frac{t}{1 + 3t^2} dt$
41.  $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$
42.  $\int \frac{(t + 1)^2}{t^2} dt$
43.  $\int \frac{x \cos(x^2)}{\sqrt{\sin(x^2)}} dx$
44.  $\int_0^\pi \cos(x + \pi) dx$
45.  $\int_0^{1/2} \cos(\pi x) dx$
46.  $\int_0^{\pi/2} e^{-\cos(\theta)} \sin(\theta) d\theta$
47.  $\int_1^2 2xe^{x^2} dx$
48.  $\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$
49.  $\int_{-1}^{e-2} \frac{1}{t + 2} dt$
50.  $\int_1^4 \frac{\cos \sqrt{x}}{\sqrt{x}} dx$
51.  $\int_0^2 \frac{x}{(1 + x^2)^2} dx$

52. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

(a)  $\int x \sin(x^2) dx$

- (b)  $\int x^2 \sin(x) dx$
- (c)  $\int \frac{x^2}{1 + x^2} dx$
- (d)  $\int \frac{x}{(1 + x^2)^2} dx$
- (e)  $\int x^3 e^{x^2} dx$
- (f)  $\int \frac{\sin(x)}{2 + \cos(x)} dx$

53. Find the exact area under the graph of  $f(x) = xe^{x^2}$  between  $x = 0$  and  $x = 2$ .

54. Find the exact area under the graph of  $f(x) = \frac{1}{x + 1}$  between  $x = 0$  and  $x = 2$ .

55. Find  $\int 4x(x^2 + 1) dx$  using two methods:

- (a) Do the multiplication first, and then antidifferentiate.
- (b) Use the substitution  $w = x^2 + 1$ .
- (c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

56. (a) Find  $\int \sin \theta \cos \theta d\theta$

(b) You probably solved part (a) by making the substitution  $w = \sin \theta$  or  $w = \cos \theta$ . (If not, go back and do it that way.) Now find  $\int \sin \theta \cos \theta d\theta$  by making the other substitution.

(c) There is yet another way of finding this integral which involves the trigonometric identities:

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta.\end{aligned}$$

Find  $\int \sin \theta \cos \theta d\theta$  using one of these identities and then the substitution  $w = 2\theta$ .

(d) You should now have three different expressions for the indefinite integral  $\int \sin \theta \cos \theta d\theta$ . Are they really different? Are they all correct? Explain.

## Substitution Integrals - Applications

57. Oil is leaking out of a ruptured tanker at the rate of  $r(t) = 50e^{-0.02t}$  thousand liters per minute.
- (a) At what rate, in liters per minute, is oil leaking out at  $t = 0$ ? At  $t = 60$ ?
- (b) How many liters leak out during the first hour?
58. If we assume that wind resistance is proportional to velocity, then the downward velocity,  $v$ , of a body of

mass  $m$  falling vertically is given by

$$v = \frac{mg}{k}(1 - e^{-kt/m})$$

where  $g$  is the acceleration due to gravity and  $k$  is a constant. Find the height of the body,  $h$ , above the surface of the earth as a function of time. Assume the body starts at height  $h_0$ .

## Integration by Parts

59. For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate. Do not evaluate the integrals.

(a)  $\int x \sin(x) dx$

(b)  $\int \frac{x^2}{1+x^3} dx$

(c)  $\int xe^{x^2} dx$

(d)  $\int x^2 \cos(x^3) dx$

(e)  $\int \frac{1}{\sqrt{3x+1}} dx$

(f)  $\int x^2 \sin x dx$

(g)  $\int \ln x dx$

To practice computing integrals by parts, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

For Questions #60 to #83, evaluate the integral.

60.  $\int t \sin t dt$

61.  $\int te^{5t} dt$

62.  $\int pe^{-0.1p} dp$

63.  $\int (z+1)e^{2z} dz$

64.  $\int \ln x dx$

65.  $\int y \ln y dy$

66.  $\int x^3 \ln x dx$

67.  $\int q^5 \ln(5q) dq$

68.  $\int t^2 \sin t dt$

69.  $\int x^2 \cos(3x) dx$

70.  $\int (\ln t)^2 dt$

71.  $\int t^2 e^{5t} dt$

72.  $\int y\sqrt{y+3} dy$

73.  $\int (t+2)\sqrt{2+3t} dt$

74.  $\int (p+1) \sin(p+1) dp$

75.  $\int \frac{z}{e^z} dz$

76.  $\int \frac{\ln x}{x^2} dx$

77.  $\int \frac{y}{\sqrt{5-y}} dy$

78.  $\int \frac{t+7}{\sqrt{5-t}} dt$

79.  $\int x(\ln x)^2 dx$

80.  $\int \arcsin(w) dw$

81.  $\int \arctan(7x) dx$

82.  $\int x \arctan(x^2) dx$

83.  $\int x^3 e^{x^2} dx$

84.  $\int_1^5 \ln t \, dt$

85.  $\int_3^5 x \cos x \, dx$

86.  $\int_0^{10} ze^{-z} \, dz$

87.  $\int_1^3 t \ln(t) \, dt$

88.  $\int_0^1 \arctan(y) \, dy$

89.  $\int_0^5 \ln(1+t) \, dt$

90.  $\int_0^1 \arcsin z \, dz$

91.  $\int_0^1 x \arcsin(x^2) \, dx$

92. Find the area under the curve  $y = te^{-t}$  on the interval  $0 \leq t \leq 2$ .

93. Find the area under the curve  $f(z) = \arctan z$  on the interval  $[0, 2]$ .

94. Use integration by parts twice to find  $\int e^x \sin(x) \, dx$ .

95. Use integration by parts twice to find  $\int e^y \cos(y) \, dy$ .

96. The concentration,  $C$ , in ng/ml, of a drug in the blood as a function of the time,  $t$ , in hours since the drug was administered is given by

$$C = 15te^{-0.2t}.$$

The area under the concentration curve is a measure of the overall effect of the drug on the body, called the *bioavailability*. Find the bioavailability of the drug between  $t = 0$  and  $t = 3$ .

97. During a surge in the demand for electricity, the rate,  $r$ , at which energy is used (in megawatts) can be approximated by

$$r = te^{-at}$$

where  $t$  is the time in hours and  $a$  is a positive constant.

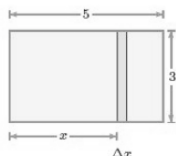
(a) Find the total energy,  $E$ , in megawatt hours used in the first  $T$  hours. Give your answer as a function of  $a$ .

(b) What happens to  $E$  as  $T \rightarrow \infty$ ?

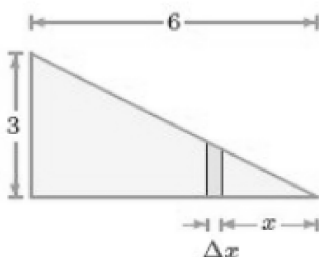
## Areas

In Questions #98-105, find the area of one strip or slice, then use that to build a definite integral representing the total area of the region. Where possible with the techniques from the class, evaluate the integral.

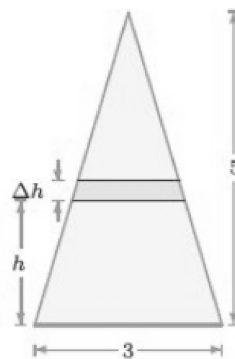
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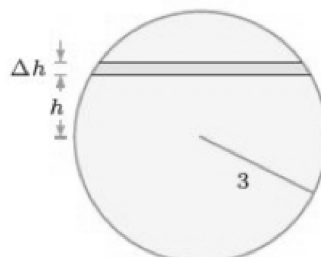
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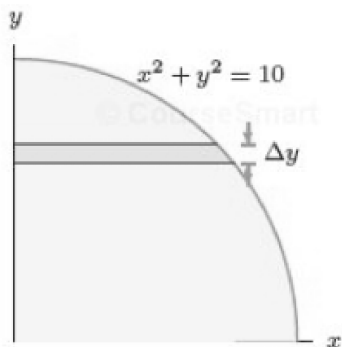
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Note: your final integral will **not** be of a type you can evaluate using substitution or by parts. **Stop once**

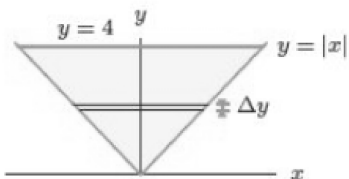
you have created the integral.

102.

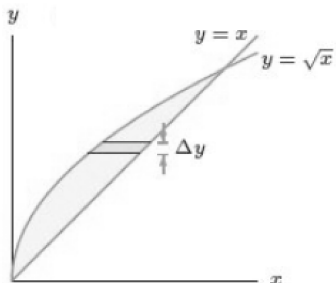


Note: your final integral will **not** be of a type you can evaluate using substitution or by parts. **Stop once you have created the integral.**

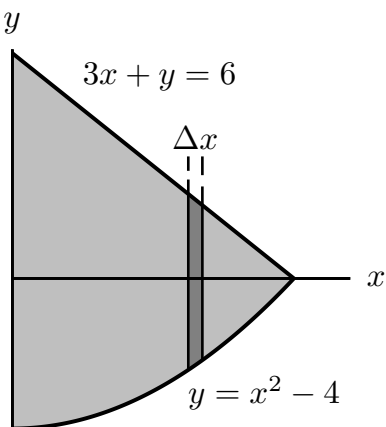
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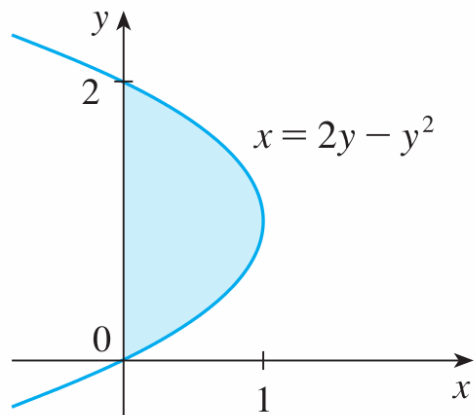
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105.



106. The area of the region that lies to the right of the  $y$ -axis and to the left of the parabola  $x = 2y - y^2$  (the shaded region in the figure) is given by the integral  $\int_0^2 (2y - y^2) dy$ . (Turn your head clockwise and think of the region as lying below the curve  $x = 2y - y^2$  from  $y = 0$  to  $y = 2$ .) Find the area of the region.



107. The boundaries of the shaded region are the  $y$ -axis, the line  $y = 1$ , and the curve  $y = \sqrt[4]{x}$ . Find the area of this region by writing  $x$  as a function of  $y$  and integrating with respect to  $y$ .

