

## Week #7 - The Fundamental Theorem of Calculus

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

### Definite Integrals in Modeling

1. The rate at which the world's oil is being consumed is continuously increasing. Suppose the rate of oil consumption (in billions of barrels per year) is given by the function  $r = f(t)$ , where  $t$  is measured in years and  $t = 0$  is the start of 2004.

- (a) Write a definite integral which represents the total quantity of oil used between the start of 2004 and the start of 2009.
- (b) Suppose  $r = 32e^{0.05t}$ . Using a left-hand sum with five subdivisions, find an approximate value for the total quantity of oil used between the start of 2004 and the start of 2009.
- (c) Interpret each of the five terms in the sum from part (b) in terms of oil consumption.

(a) Quantity used =  $\int_0^5 f(t) dt$ .

(b) Using a left sum, our approximation is

$$32e^{0.05(0)} + 32e^{0.05(1)} + 32e^{0.05(2)} + 32e^{0.05(3)} + 32e^{0.05(4)} = 177.270.$$

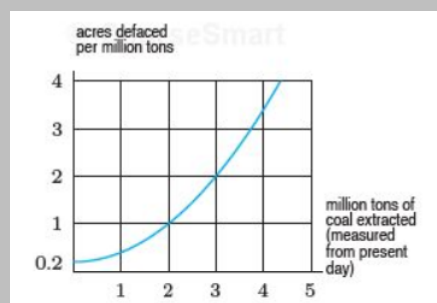
Since  $f$  is an increasing function, the left endpoint values of  $f$  are lower than anywhere else on the subintervals, so this represents an underestimate.

(c) Each term is a lower estimate of one year's consumption of oil.

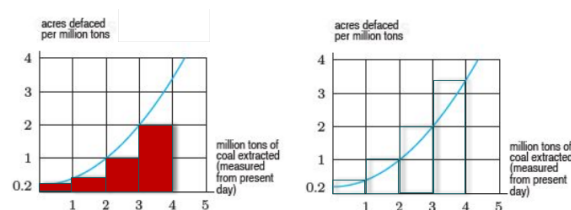
Problem code: RNXMK

2. As coal deposits are depleted, it becomes necessary to strip-mine larger areas for each ton of coal. The graph below shows the number of acres of land per million tons of coal that will be defaced during strip-mining as a function of the number of million tons removed, starting from the present day.

- (a) Estimate the total number of acres defaced in extracting the next 4 million tons of coal (measured from the present day). Draw four rectangles under the curve, and compute their area.
- (b) Re-estimate the number of acres defaced using rectangles above the curve.
- (c) Combine your answers to parts (a) and (b) to get a better estimate of the actual number of acres defaced.



(a)



Using rectangles under the curve, we get

$$\text{Acres defaced} \approx (1)(0.2 + 0.4 + 1 + 2) = 3.6 \text{ acres.}$$

(b) Using rectangles above the curve, we get

$$\text{Acres defaced} \approx (1)(0.4 + 1 + 2 + 3.5) = 6.9 \text{ acres.}$$

(c) The number of acres defaced is between 3.6 and 6.9, so we estimate the average, 5.25 acres.

Problem code: LFPKD

3. The following table gives the emissions,  $E$ , of nitrogen oxides, in millions of metric tons per year, in the US. Let  $t$  be the number of years since 1970 and  $E = f(t)$ .

(a) What are the units and meaning of

$$\int_0^{30} f(t) dt?$$

(b) Estimate  $\int_0^{30} f(t) dt$ .

Year	1970	1975	1980	1985	1990	1995	2000
E	26.9	26.4	27.1	25.8	25.5	25.0	22.6

(a) The integral  $\int_0^{30} f(t) dt$  represents the total emissions of nitrogen oxides, in millions of metric tons, during the period 1970 to 2000.

(b) We estimate the integral using left- and right-hand sums:

$$\begin{aligned} \text{Left sum} &= (26.9)5 + (26.4)5 + (27.1)5 + (25.8)5 \\ &\quad + (25.5)5 + (25.0)5 \\ &= 783.5. \end{aligned}$$

$$\begin{aligned} \text{Right sum} &= (26.4)5 + (27.1)5 + (25.8)5 + (25.5)5 \\ &\quad + (25.0)5 + (22.6)5 \\ &= 762.0. \end{aligned}$$

We average the left- and right-hand sums to find an improved estimate of the integral:

$$\int_0^{30} f(t) dt \approx \frac{783.5 + 762.0}{2} = 772.8 \text{ million metric tons.}$$

Between 1970 and 2000, about 772.8 million metric tons of nitrogen oxides were emitted.

Problem code: DAJZM

4. Coal gas is produced at a gasworks. Pollutants in the gas are removed by scrubbers, which become less and less efficient as time goes on. The following measurements, made at the start of each month, show the rate at which pollutants are escaping (in tons/month) in the gas:

Time (months)	0	1	2	3	4	5	6
Rate pollutants escape (tons/month)	5	7	8	10	13	16	20

(a) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the **first month**.

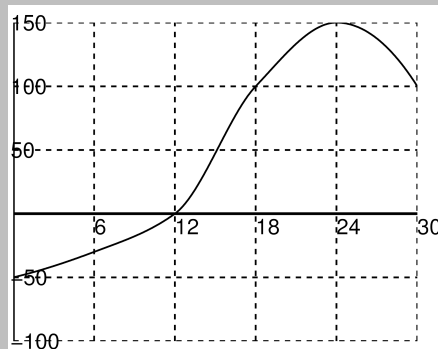
(b) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the **six months** shown in the table.

(a) An overestimate is  $(7 \text{ tons/month}) \cdot (1 \text{ month}) = 7 \text{ tons}$ . An underestimate is  $(5 \text{ tons/month}) \cdot (1 \text{ month}) = 5 \text{ tons}$ .

(b) An overestimate is  $7+8+10+13+16+20 = 74 \text{ tons}$ . An underestimate is  $5+7+8+10+13+16 = 59 \text{ tons}$ .

Problem code: HXHLU

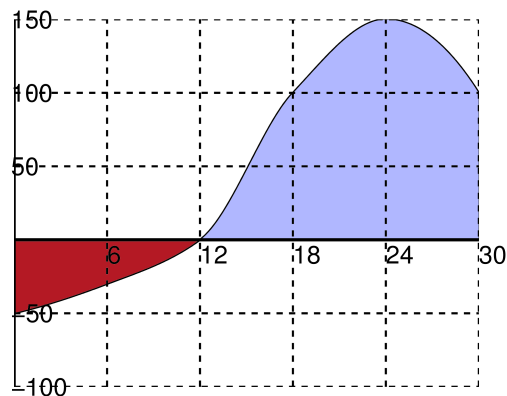
5. The graph below shows the rate of change of the quantity of water in a water tower, in liters per day, during the month of April. If the tower had 12,000 liters of water in it on April 1, estimate the quantity of water in the tower on April 30.



Suppose  $V(t)$  represents the total quantity of water in the water tower at time  $t$ , where  $t$  is in days since April 1. Then the graph shown in the problem is a graph of the volume's rate of change, or  $\frac{dV}{dt}$  or  $V'(t)$ . By the Fundamental Theorem, the change in the volume over the 30 days of April is given by:

$$V(30) - V(0) = \int_0^{30} V'(t) dt.$$

This relationship means we can calculate the change in the volume of water by calculating the area under the curve. Each box represents about  $(50 \text{ l/d}) \cdot (6 \text{ d}) = 300$  liters.



In the red region, from  $t = 0$  to  $t = 12$ , the graph is below the  $t$  axis, indicating that water is being lost from the tower. There is a little more than one box, but for simplicity we'll call it one square, so roughly  $\approx 300$  liters were lost.

From  $t = 12$  to  $t = 30$  (blue region), there are between

6 and 7 boxes in total (let's say 6 for simplicity), or roughly +1800 liters.

Thus,

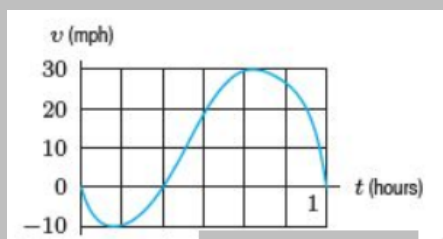
$$\int_0^{30} V'(t) dt \approx 1800 - 300 = 1500$$

This is the net *change* in the volume of water. Solving for  $V(30)$ , the *amount* of water on April 30th,

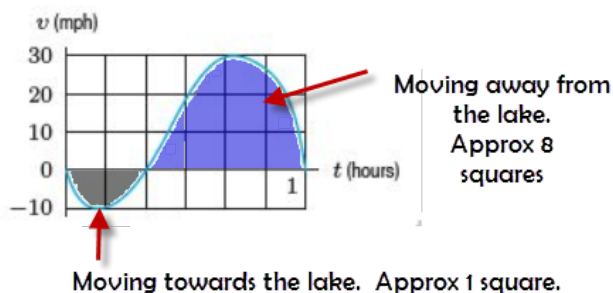
$$\begin{aligned} V(30) &= V(0) + \int_0^{30} V'(t) dt \\ &= 12,000 + 1500 = 13,500 \text{ liters.} \end{aligned}$$

Problem code: JGPQP

6. A bicyclist pedals along a straight road with velocity  $v$  given in the graph below. She starts 5 miles from a lake; positive velocities take her away from the lake and negative velocities take her toward the lake. When is the cyclist farthest from the lake, and how far away is she then?



Notice that the area of a square on the graph represents  $(10 \text{ mph}) \cdot (1/6) \text{ h} = (5/3) \text{ miles}$ .



At  $t = 1/3$  hours,  $v = 0$ . The area between the curve  $v$  and the  $t$ -axis over the interval  $0 \leq t \leq 1/3$  is  $-\int_0^{1/3} v dt \approx$  one square  $= 5/3$  miles. Since  $v$  is negative here, she is moving toward the lake. Since she starts 5 miles from the lake at time  $t = 0$ , at  $t = 1/3$  she is about  $5 - 5/3 = 10/3$  miles from the lake (so closer than when she started).

For  $1/3 \leq t \leq 1$ ,  $v$  is positive, so she is moving away from the lake. Her *change* in distance from the lake between the times  $t = 0$  and the end point  $t = 1$  is

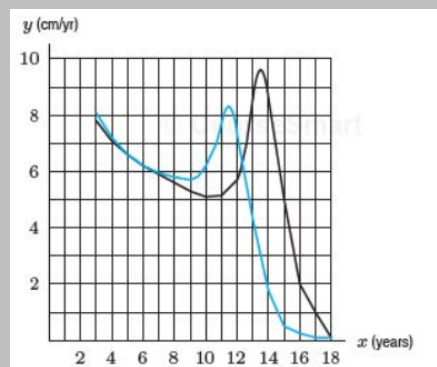
given by

$$\begin{aligned} \int_0^1 v dt &= \int_0^{1/3} v dt + \int_{1/3}^1 v dt \\ &\approx -\frac{5}{3} + 8 \cdot \frac{5}{3} = \frac{35}{3} = 11.667 \text{ miles,} \end{aligned}$$

Relative to her starting point of 5 miles at  $t = 0$ , the cyclist is about  $5 + 35/3 = 50/3 = 16.667$  miles from the lake at  $t = 1$ . Since, starting from the moment  $t = 1/3$ , she is always moving away from the lake, the cyclist will be farthest from the lake at the latest possible time,  $t = 1$ . The maximal distance at that time equals  $50/3 = 16.667$  miles.

Problem code: BLKMF

7. Height velocity graphs are used by endocrinologists (doctors specializing in the study of hormones) to follow the progress of children with growth deficiencies. The graph below shows the height velocity curves of an average boy and an average girl between ages 3 and 18.



- Which curve is for girls and which is for boys? Explain how you can tell.
  - About how much does the average boy grow between ages 3 and 10?
  - The growth spurt associated with adolescence and the onset of puberty occurs between ages 12 and 15 for the average boy and between ages 10 and 12.5 for the average girl. Estimate the height gained by each average child during this growth spurt.
  - When fully grown, about how much taller is the average man than the average woman? (The average boy and girl are about the same height at age 3.)
- (a) Since men are generally taller than women, the curve with the larger area under it is the height velocity of the boys. The area under each curve represents the change in growth in centimeters. Therefore, the black curve is for boys, the lighter grey one for girls.

- (b) Each square below the height velocity curve has area  $(1 \text{ cm/yr}) \cdot 1 \text{ yr} = 1 \text{ cm}$ . Counting squares lying below the black curve gives about 43 cm. Thus, on average, boys grow about 43 cm between ages 3 and 10.
- (c) Counting squares lying below the black curve gives about 23 cm growth for boys during their growth spurt. Counting squares lying below the coloured curve gives about 18 cm for girls during their growth spurt.

- (d) We can measure the difference in growth by counting squares that lie between the two curves. Between ages 2 and 12.5, the average girl grows faster than the average boy. Counting squares yields about 5 cm between the coloured and black curves for  $2 \leq x \leq 12.5$ . Counting squares between the curves for  $12.5 \leq x \leq 18$  gives about 18 squares. Thus, there is a net increase of boys over girls by about  $18 - 5 = 13 \text{ cm}$ .

Problem code: MMNBS

## Integration By Anti-Derivatives

To practice computing anti-derivatives, do as many of the problems from below, **and from the textbook**, as you need.

Additional problems. Evaluate the following integrals.

$$8. \int_{-1}^2 (x^3 - 2x) dx$$

$$\int_{-1}^2 (x^3 - 2x) dx = \left[ \frac{x^4}{4} - x^2 \right]_{-1}^2 = \left( \frac{2^4}{4} - 2^2 \right) - \left( \frac{(-1)^4}{4} - (-1)^2 \right) = (4 - 4) - \left( \frac{1}{4} - 1 \right) = 0 - \left( -\frac{3}{4} \right) = \frac{3}{4}$$

Problem code: EEDGH ([Video Solution by C.C.](#))

$$9. \int_{-1}^1 x^{100} dx$$

$$\int_{-1}^1 x^{100} dx = \left[ \frac{1}{101} x^{101} \right]_{-1}^1 = \frac{1}{101} - \left( -\frac{1}{101} \right) = \frac{2}{101}$$

Problem code: BLYFQ

$$10. \int_1^4 (5 - 2t + 3t^2) dt$$

$$\int_1^4 (5 - 2t + 3t^2) dt = [5t - t^2 + t^3]_1^4 = (20 - 16 + 64) - (5 - 1 + 1) = 68 - 5 = 63$$

Problem code: EFXPW

$$11. \int_0^1 \left( 1 + \frac{1}{2}u^4 - \frac{2}{5}u^9 \right) du$$

$$\int_0^1 \left( 1 + \frac{1}{2}u^4 - \frac{2}{5}u^9 \right) du = \left[ u + \frac{1}{10}u^5 - \frac{1}{25}u^{10} \right]_0^1 = \left( 1 + \frac{1}{10} - \frac{1}{25} \right) - 0 = \frac{50+5-2}{50} = \frac{53}{50}$$

Problem code: DGVTF

$$12. \int_1^9 \sqrt{x} dx$$

$$\int_1^9 \sqrt{x} dx = \int_1^9 x^{1/2} dx = \left[ \frac{x^{3/2}}{3/2} \right]_1^9 = \frac{2}{3} \left[ x^{3/2} \right]_1^9 = \frac{2}{3} (9^{3/2} - 1^{3/2}) = \frac{2}{3} (27 - 1) = \frac{52}{3}$$

Problem code: VRPFA

$$13. \int_1^8 x^{-2/3} dx$$

$$\int_1^8 x^{-2/3} dx = \left[ \frac{x^{1/3}}{1/3} \right]_1^8 = 3 \left[ x^{1/3} \right]_1^8 = 3(8^{1/3} - 1^{1/3}) = 3(2 - 1) = 3$$

Problem code: RPGSB

$$14. \int_{\pi/6}^{\pi} \sin \theta d\theta$$

$$\int_{\pi/6}^{\pi} \sin \theta d\theta = [-\cos \theta]_{\pi/6}^{\pi} = -\cos \pi - (-\cos \frac{\pi}{6}) = -(-1) - (-\sqrt{3}/2) = 1 + \sqrt{3}/2$$

Problem code: KVEEJ

$$15. \int_{-5}^5 e dx$$

The number  $e$  is just a constant.

$$\int_{-5}^5 e dx = [ex]_{-5}^5 = 5e - (-5e) = 10e$$

Problem code: NKXLD

$$16. \int_0^1 (u+2)(u-3) du$$

You need to expand the product before you can integrate.

$$\int_0^1 (u+2)(u-3) du = \int_0^1 (u^2 - u - 6) du = \left[ \frac{1}{3}u^3 - \frac{1}{2}u^2 - 6u \right]_0^1 = \left( \frac{1}{3} - \frac{1}{2} - 6 \right) - 0 = -\frac{37}{6}$$

Problem code: TCLXX

$$17. \int_0^4 (4-t)\sqrt{t} dt$$

To evaluate this integral, you need to expand the product first; you can't integrate the product  $(4-t)\sqrt{t}$  as it is originally written.

$$\int_0^4 (4-t)\sqrt{t} dt = \int_0^4 (4-t)t^{1/2} dt = \int_0^4 (4t^{1/2} - t^{3/2}) dt = \left[ \frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2} \right]_0^4 = \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{320 - 192}{15} = \frac{128}{15}$$

Problem code: QKTST

$$18. \int_1^9 \frac{x-1}{\sqrt{x}} dx$$

$$\int_1^9 \frac{x-1}{\sqrt{x}} dx = \int_1^9 \left( \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}} \right) dx = \int_1^9 (x^{1/2} - x^{-1/2}) dx = \left[ \frac{2}{3}x^{3/2} - 2x^{1/2} \right]_1^9 = \left( \frac{2}{3} \cdot 27 - 2 \cdot 3 \right) - \left( \frac{2}{3} - 2 \right) = 12 - \left( -\frac{4}{3} \right) = \frac{40}{3}$$

Problem code: WFDYH

$$19. \int_0^2 (y-1)(2y+1) dy$$

$$\int_0^2 (y-1)(2y+1) dy = \int_0^2 (2y^2 - y - 1) dy = \left[ \frac{2}{3}y^3 - \frac{1}{2}y^2 - y \right]_0^2 = \left( \frac{16}{3} - 2 - 2 \right) - 0 = \frac{4}{3}$$

Problem code: FYWCS

$$20. \int_0^{\pi/4} \sec^2 t \, dt$$

$$\int_0^{\pi/4} \sec^2 t \, dt = [\tan t]_0^{\pi/4} = \tan \frac{\pi}{4} - \tan 0 = 1 - 0 = 1$$

Problem code: GNWVP

$$21. \int_0^{\pi/4} \sec \theta \tan \theta \, d\theta$$

$$\int_0^{\pi/4} \sec \theta \tan \theta \, d\theta = [\sec \theta]_0^{\pi/4} = \sec \frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$$

Problem code: HMCVE

$$22. \int_1^2 (1 + 2y)^2 \, dy$$

$$\int_1^2 (1 + 2y)^2 \, dy = \int_1^2 (1 + 4y + 4y^2) \, dy = [y + 2y^2 + \frac{4}{3}y^3]_1^2 = (2 + 8 + \frac{32}{3}) - (1 + 2 + \frac{4}{3}) = \frac{62}{3} - \frac{13}{3} = \frac{49}{3}$$

Problem code: ZANXV

$$23. \int_0^3 (2 \sin x - e^x) \, dx$$

$$\int_0^3 (2 \sin x - e^x) \, dx = [-2 \cos x - e^x]_0^3 = (-2 \cos 3 - e^3) - (-2 - 1) = 3 - 2 \cos 3 - e^3$$

Problem code: MQWPE

$$24. \int_1^2 \frac{v^3 + 3v^6}{v^4} \, dv$$

$$\int_1^2 \frac{v^3 + 3v^6}{v^4} \, dv = \int_1^2 \left( \frac{1}{v} + 3v^2 \right) \, dv = [\ln |v| + v^3]_1^2 = (\ln 2 + 8) - (\ln 1 + 1) = \ln 2 + 7$$

Problem code: GJNDH

$$25. \int_1^{18} \sqrt{\frac{3}{z}} \, dz$$

$$\int_1^{18} \sqrt{\frac{3}{z}} \, dz = \int_1^{18} \sqrt{3} z^{-1/2} \, dz = \sqrt{3} [2z^{1/2}]_1^{18} = 2\sqrt{3}(18^{1/2} - 1^{1/2}) = 2\sqrt{3}(3\sqrt{2} - 1)$$

Problem code: BGASB

$$26. \int_0^1 (x^e + e^x) \, dx$$

$$\int_0^1 (x^e e^x) \, dx = \left[ \frac{x^{e+1}}{e+1} + e^x \right]_0^1 = \left( \frac{1}{e+1} + e \right) - (0 + 1) = \frac{1}{e+1} + e - 1$$

Problem code: CXAAN

$$27. \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} \, dx$$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} \, dx = [8 \arctan x]_{1/\sqrt{3}}^{\sqrt{3}} = 8 \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = 8 \left( \frac{\pi}{6} \right) = \frac{4\pi}{3}$$

Problem code: ZKTDL ([Video Solution by J.G.](#))

$$28. \int_1^2 \frac{4+u^2}{u^3} du$$

$$\int_1^2 \frac{4+u^2}{u^3} du = \int_1^2 (4u^{-3} + u^{-1}) du = \left[ \frac{4}{-2} u^{-2} + \ln |u| \right]_1^2 = \left[ \frac{-2}{u^2} + \ln u \right]_1^2 = \left( -\frac{1}{2} + \ln 2 \right) - \left( -2 + \ln 1 \right) = \frac{3}{2} + \ln 2$$

Problem code: XLPDT

$$29. \int_{-1}^1 e^{u+1} du$$

$$\int_{-1}^1 e^{u+1} du = [e^{u+1}]_{-1}^1 = e^2 - e^0 = e^2 - 1 \text{ [or start with } e^{u+1} = e^u e^1]$$

Problem code: NKMBJ

$$30. \int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$$

$$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx = [4 \arcsin x]_{1/2}^{1/\sqrt{2}} = 4 \left( \frac{\pi}{4} - \frac{\pi}{6} \right) = 4 \left( \frac{\pi}{12} \right) = \frac{\pi}{3}$$

Problem code: TWKNW

$$31. \int_0^\pi f(x) dx \text{ where } f(x) = \begin{cases} \sin x & \text{if } 0 \leq x < \pi/2 \\ \cos x & \text{if } \pi/2 \leq x \leq \pi \end{cases}$$

$$\text{If } f(x) = \begin{cases} \sin x & \text{if } 0 \leq x < \pi/2 \\ \cos x & \text{if } \pi/2 \leq x \leq \pi \end{cases} \text{ then}$$

$$\int_0^\pi f(x) dx = \int_0^{\pi/2} \sin x dx + \int_{\pi/2}^\pi \cos x dx = [-\cos x]_0^{\pi/2} + [\sin x]_{\pi/2}^\pi = -\cos \frac{\pi}{2} + \cos 0 + \sin \pi - \sin \frac{\pi}{2} = -0 + 1 + 0 - 1 = 0$$

Problem code: CWDJK

$$32. \int_{-2}^2 f(x) dx \text{ where } f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0 \\ 4-x^2 & \text{if } 0 < x \leq 2 \end{cases}$$

$$\text{If } f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0 \\ 4-x^2 & \text{if } 0 < x \leq 2 \end{cases} \text{ then}$$

$$\int_{-2}^2 f(x) dx = \int_{-2}^0 2 dx + \int_0^2 (4-x^2) dx = [2x]_{-2}^0 + [4x - \frac{1}{3}x^3]_0^2 = [0 - (-4)] + (\frac{16}{3} - 0) = \frac{28}{3}$$

Problem code: TEKFB

33. If  $w'(t)$  is the rate of growth of a child in pounds per year, what does  $\int_5^{10} w'(t) dt$  represent?

If  $w'(t)$  is the rate of change of weight in pounds per year, then  $w(t)$  represents the weight in pounds of the child at age  $t$ . We know from the Net Change Theorem that  $\int_5^{10} w'(t) dt = w(10) - w(5)$ , so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

Problem code: YFAZP

34. The current in a wire is defined as the derivative of the charge:  $I(t) = Q'(t)$ . What does  $\int_a^b I(t) dt$  represent?

$\int_a^b I(t) dt = \int_a^b Q'(t) dt = Q(b) - Q(a)$  by the Net Change Theorem, so it represents the charge  $Q$  from time  $t = a$  to  $t = b$ .

Problem code: KTCGT

35. If oil leaks from a tank at a rate of  $r(t)$  gallons per minute at time  $t$ , what does  $\int_0^{120} r(t)dt$  represent?

Since  $r(t)$  is the rate at which oil leaks, we can write  $r(t) = -V'(t)$ , where  $V(t)$  is the volume of oil at time  $t$ . [Note that the minus sign is needed because  $V$  is decreasing, so  $V'(t)$  is negative, but  $r(t)$  is positive.] Thus, by the Net Change Theorem,  $\int_0^{120} r(t)dt = -\int_0^{120} V'(t)dt = -[V(120) - V(0)] = V(0) - V(120)$ , which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

Problem code: ECJBU

36. A honeybee population starts with 100 bees and increases at a rate of  $n'(t)$  bees per week. What does  $100 + \int_0^{15} n'(t)dt$  represent?

By the Net Change Theorem,  $\int_0^{15} n'(t)dt = n(15) - n(0) - 100$  represents the increase in the bee population in 15 weeks. So  $100 + \int_0^{15} n'(t)dt = n(15)$  represents the total bee population after 15 weeks.

Problem code: RZBBF

37. The marginal revenue function  $R'(x)$  is defined as the derivative of the revenue function  $R(x)$ , where  $x$  is the number of units sold. What does  $\int_{1000}^{5000} R'(x)dx$  represent?

By the Net Change Theorem,  $\int_{1000}^{5000} R'(x)dx = R(5000) - R(1000)$ , so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

Problem code: DNGBN

38. If  $f(x)$  is the slope of a trail at a distance of  $x$  miles from the start of the trail, what does  $\int_3^5 f(x)dx$  represent?

The slope of the trail is the rate of change of the elevation  $E$ , so  $f(x) = E'(x)$ . By the Net Change Theorem,  $\int_3^5 f(x)dx = \int_3^5 E'(x)dx = E(5) - E(3)$  is the change in elevation  $E$  between  $x = 3$  miles and  $x = 5$  miles from the start of the trail.

Problem code: XFNGW

39. If  $x$  is measured in meters and  $f(x)$  is measured in Newtons, what are the units for  $\int_0^{100} f(x)dx$ ?

In general, the unit of measurement for  $\int_a^b f(x)dx$  is the product of the unit for  $f(x)$  and the unit for  $x$ . Since  $f(x)$  is measured in newtons and  $x$  is measured in meters, the units for  $\int_0^{100} f(x)dx$  are newton-meters. (A newton-meter is abbreviated  $N \cdot m$ .)

Problem code: TWZAP

40. The linear density of a rod of length 4m is given by  $\rho(x) = 9 + 2\sqrt{x}$  measured in kilograms per meter, where  $x$  is measured in meters from one end of the rod. Find the total mass of the rod.

Since  $m'(x) = \rho(x)$ ,  $m = \int_0^4 \rho(x)dx = \int_0^4 (9 + 2\sqrt{x})dx = [9x + \frac{4}{3}x^{3/2}]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}\text{kg}$

Problem code: MMGHN

41. Water flows from the bottom of a storage tank at a rate of  $r(t) = 200 - 4t$  liters per minute, where  $0 \leq t \leq 50$ . Find the amount of water that flows from the tank during the first 10 minutes.

By the Net Change Theorem, the amount of water that flows from the tank during the first 10 minutes is  $\int_0^{10} r(t)dt = \int_0^{10} (200 - 4t)dt = [200t - 2t^2]_0^{10} = (2000 - 200) - 0 = 1800$  liters.

Problem code: JKBEJ

42. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule on 5 intervals to estimate the distance travelled by the car.

$t(\text{s})$	$v(\text{mi / h})$	$t(\text{s})$	$v(\text{mi / h})$
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

Let  $s$  be the position of the car. We know from the Net Change theorem, or the integral relationship between velocity and position, that  $s(100) - s(0) = \int_0^{100} v(t)dt$ . We use the Midpoint rule for  $0 \leq t \leq 100$  with  $n = 5$ . Note that the length of each of the five time intervals is 20 seconds  $= \frac{20}{3600}$  hour  $= \frac{1}{180}$  hour. So the distance travelled is  $\int_0^{100} v(t)dt \approx \frac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \frac{1}{180}(38 + 58 + 51 + 53 + 47) = \frac{247}{180} \approx 1.4$  miles.

Problem code: GCSTZ

43. Suppose that a volcano is erupting and readings of the rate  $r(t)$  at which solid materials are spewed into the atmosphere are given in the table. The time  $t$  is measured in seconds and the units for  $r(t)$  are tonnes (metric tons) per second.

$t$	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

- (a) Use the LEFT and RIGHT sums to give upper and lower estimates for the total quantity  $Q(6)$  of erupted materials after 6 seconds.
- (b) Use the Midpoint Rule to estimate  $Q(6)$ .

- (a) By the Net Change Theorem, the total amount spewed into the atmosphere is  $Q(6) - Q(0) = \int_0^6 r(t)dt = Q(6)$  since  $Q(0) = 0$ . The rate  $r(t)$  is positive, so  $Q$  is an increasing function. Thus, an upper estimate for  $Q(6)$  is  $R_6$  and a lower estimate for  $Q(6)$  is  $L_6$ .  $\Delta t = \frac{b-a}{n} = \frac{6-0}{6} = 1$ .

$$R_6 = \sum_{i=1}^6 r(t_i)\Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230 \text{ tonnes.}$$

$$L_6 = \sum_{i=1}^6 r(t_{i-1})\Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172 \text{ tonnes.}$$

- (b)  $\Delta t = \frac{b-a}{n} = \frac{6-0}{3} = 2$ .  $Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$  tonnes.

Problem code: UBFQS

44. The marginal cost of manufacturing  $x$  yards of a certain fabric is  $C'(x) = 3 - 0.01x + 0.000006x^2$  (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is  $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)dx$ .

$$\int_{2000}^{4000} C'(x)dx = \int_{2000}^{4000} (3 - 0.01x + 0.000006x^2)dx = [3x - 0.005x^2 + 0.000002x^3]_{2000}^{4000} = 60000 - 2000 = \$58000$$

Problem code: RLCCQ