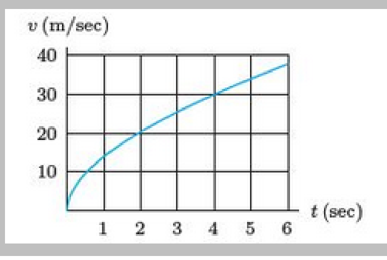
Week #6 - Defining and Estimating Integrals as Areas

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Distance And Velocity

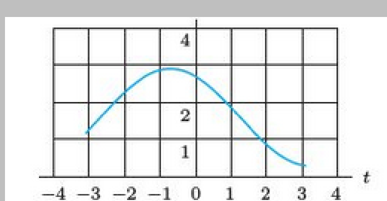
1. The graph below shows the velocity, v, of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$, by counting squares or parts of squares, in the diagram.



Just counting the squares (each of which has area representing 10 (m/s)·(s) = 10 m of distance), and allowing for the partial squares, we can see that the area under the curve from $t = 0$ to $t = 6$ is between 140 and 150 units. Therefore the distance traveled is between 140 and 150 meters.

Problem code: ALJYM

2. The figure below shows the velocity of a particle, in cm/sec, along the t-axis for $-3 \leq t \leq 3$ (t in seconds).
   (a) Describe the motion in words. Is the particle changing direction or always moving in the same direction? Is the particle speeding up or slowing down?
   (b) Make over- and underestimates of the distance traveled for $-3 \leq t \leq 3$, by counting squares or parts of squares, in the diagram.



(a) The velocity is always positive, so the particle is moving in the same direction throughout. However, the particle is speeding up until shortly before $t = 0$, and slowing down thereafter.

(b) The distance traveled is represented by the area under the curve. Using whole grid squares, we can overestimate the area as 3+3+3+3+2+1 = 15 squares, and we can underestimate the area as 1+2+2+1+0+0= 6 squares. Each square represents 1 (cm/sec)·(s) = 1 cm, so the particle moved in one direction between 6 and 15 cm.

Problem code: SUVCW

3. Consider the following table of values for $f(t)$.

t	15	17	19	21	23
$f(t)$	10	13	18	20	30

   (a) If we divide the time interval into $n = 4$ sub-intervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?
   (b) Find the left and right sums using $n = 4$.
   (c) If we divide the time interval into $n = 2$ sub-intervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?
   (d) Find the left and right sums using $n = 2$.

(a) With $n = 4$, we have $\Delta t = 2$. Then $t_0 = 15, t_1 = 17, t_2 = 19, t_3 = 21, t_4 = 23$, and $f(t_0) = 10, f(t_1) = 13, f(t_2) = 18, f(t_3) = 20, f(t_4) = 30$.

(b)

$$\begin{aligned}\text{Left sum} &= (10)(2) + (13)(2) + (18)(2) + (20)(2) \\ &= 122 \\ \text{Right sum} &= (13)(2) + (18)(2) + (20)(2) + (30)(2) \\ &= 162\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 4$. Then $t_0 = 15; t_1 = 19; t_2 = 23$ and $f(t_0) = 10; f(t_1) = 18; f(t_2) = 30$.

(d) This time, $n = 2$ intervals, so our time interval is longer at $\Delta t = 4$.

$$\begin{aligned}\text{Left sum} &= (10)(4) + (18)(4) = 112 \\ \text{Right sum} &= (18)(4) + (30)(4) = 192\end{aligned}$$

Problem code: XDQVJ

4. Consider the following table of values for $f(t)$.

t	0	4	8	12	16
$f(t)$	25	23	22	20	17

(a) If we divide the time interval into $n = 4$ subintervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?

(b) Find the left and right sums using $n = 4$.

(c) If we divide the time interval into $n = 2$ subintervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?

(d) Find the left and right sums using $n = 2$.

(a) With $n = 4$, we have $\Delta t = 4$. Then $t_0 = 0; t_1 = 4; t_2 = 8; t_3 = 12; t_4 = 16$ and $f(t_0) = 25; f(t_1) = 23; f(t_2) = 22; f(t_3) = 20; f(t_4) = 17$

(b)

$$\begin{aligned}\text{Left sum} &= (25)(4) + (23)(4) + (22)(4) + (20)(4)\\ &= 360\\ \text{Right sum} &= (23)(4) + (22)(4) + (20)(4) + (17)(4)\\ &= 328\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 8$. Then $t_0 = 0; t_1 = 8; t_2 = 16$ and $f(t_0) = 25; f(t_1) = 22; f(t_2) = 17$

(d)

$$\begin{aligned}\text{Left sum} &= (25)(8) + (22)(8) = 376\\ \text{Right sum} &= (22)(8) + (17)(8) = 312\end{aligned}$$

Problem code: BVVWR

5. At time t, in seconds, your velocity, v, in meters/ second, is given by

$$v(t) = 1 + t^2 \text{ for } 0 \le t \le 6.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the upper and lower estimates, and then average the two.

Using $\Delta t = 2$,

$$\begin{aligned}\text{Lower estimate} &= v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2\\ &= 1(2) + 5(2) + 17(2)\\ &= 46\\ \text{Upper estimate} &= v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2\\ &= 5(2) + 17(2) + 37(2)\\ &= 118\\ \text{Average} &= \frac{46+118}{2}\\ &= 82\\ \text{Distance traveled} &\approx 82\left(\frac{\text{m}}{\text{s}}\right)(\text{s}) = 82 \text{ meters.}\end{aligned}$$

Problem code: QTJWZ

6. For time, t, in hours, $0 \le t \le 1$, a bug is crawling at a velocity, v, in meters/ hour given by

$$v = \frac{1}{1+t}.$$

Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

Using $\Delta t = 0.2$, our upper estimate is comes from using the smallest possible t value on each interval, because that leads to the largest velocity. I.e. we use $t_0 = 0$, $t_1 = 0.2$, etc.

$$\underbrace{\frac{1}{1+0}}_{v(t_0)}(0.2) + \frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2) \approx 0.75$$

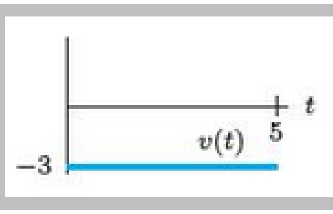
The lower estimate is

$$\frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2) + \frac{1}{1+1}(0.2) \approx 0.65$$

Since v is a decreasing function, the bug has crawled more than 0.65 meters, but less than 0.75 meters. We average the two to get a better estimate: $0.65 + 0.75$ $2 = 0.70$ meters:

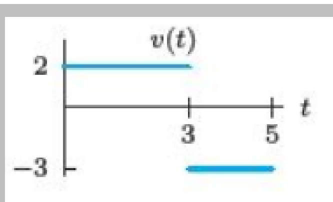
Problem code: UQBQD

For questions 7 to 10, the graph shows the velocity, in cm/sec, of a particle moving along the x-axis. Compute the particle's change in position, left (negative) or right (positive), between times $t = 0$ and $t = 5$ seconds, using the area interpretation of the integral.

7. 

The velocity is constant and negative, so the change in position is $-3 \cdot 5$ cm, that is 15 cm to the left.

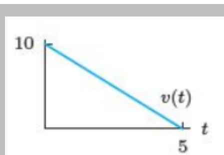
Problem code: GKNDR

8. 

From $t = 0$ to $t = 3$, the velocity is constant and positive, so the change in position is $2 \cdot 3$ cm, that is 6 cm to the right.
From $t = 3$ to $t = 5$, the velocity is negative and constant, so the change in position is $3 \cdot 2$ cm, that is 6 cm to the left.
Thus the total change in position is 0. The particle moves 6 cm to the right, followed by 6 cm to the left, and returns to where it started.

Problem code: LYBZU

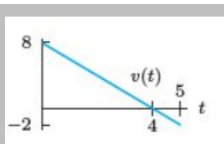
9.



From $t = 0$ to $t = 5$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)5 \cdot 10 = 25$. Thus the change in position is 25 cm to the right.

Problem code: WXKGB

10.



From $t = 0$ to $t = 4$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)4 \cdot 8 = 16$.

Thus the change in position is 16 cm to the right for$t = 0$ to $t = 4$.

From $t = 4$ to $t = 5$, the velocity is negative so the change in position is to the left. The distance traveled to the left is given by the area of the triangle, $(1/2)bh = (1/2)1 \cdot 2 = 1$.

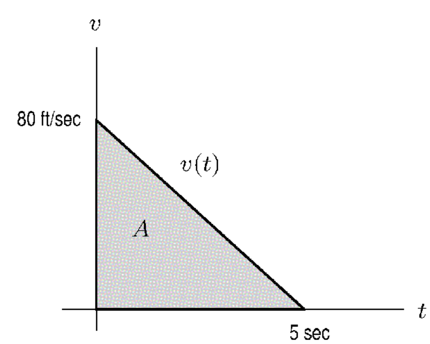
Thus the total change in position is $16 - 1 = 15$ cm to the right.

Problem code: TLTBA

11. A car going 80 ft/s (about 90 km/h) brakes to a stop in five seconds. Assume the deceleration is constant.

   (a) Graph the velocity against time, t, for $0 \le t \le 5$ seconds.
   (b) Represent, as an area on the graph, the total distance traveled from the time the brakes are applied until the car comes to a stop.
   (c) Find this area and hence the distance traveled.

(a)



(b) The total distance is represented by the shaded region A, the area under the graph of $v(t)$.

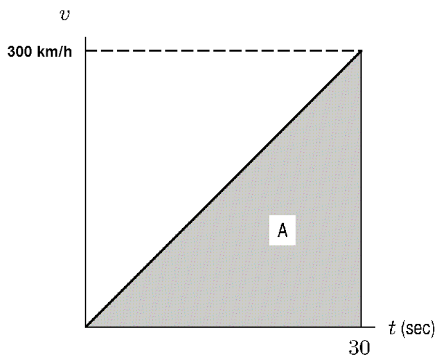
(c) A is a triangle, so its area is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(5 \text{ sec})(80 \text{ ft/s}) = 200 \text{ ft}$$

Problem code: HRZYT

12. A 727 jet needs to attain a speed of 320 km/h to take off. If it can accelerate from 0 to 320 km/h in 30 seconds, how long must the runway be? (Assume constant acceleration.)

The graph of the velocity of the plane is shown below:



The distance travelled will be represented by the area under the velocity graph. However, it is helpful to first convert everything to meters and seconds:
320 km/h $= \frac{1000}{60\cdot 60}320$ m/s ≈ 88.89 m/s.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(30 \text{ sec})(88.89 \text{ m/s}) \approx 1333.3\text{m}$$

The plane requires a roughly 1.3 km long runway to take off.

Problem code: VNYUS

13. A student is speeding down Route 11 in his fancy red Porsche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him. The "black box" in the Porsche records the car's speed every two seconds, producing the following table. The speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.

Time since brakes applied (sec)	0	2	4	6	8	10
Speed (ft/sec)	100	80	50	25	10	0

(a) What is your best estimate of the total distance the student's car traveled before coming to rest?

(b) Which one of the following statements can you justify from the information given?

(i) The car stopped before getting to the skunk.

(ii) The "black box" data is inconclusive. The skunk may or may not have been hit.

(iii) The skunk was hit by the car.

To find the distance the car moved before stopping, we estimate the distance traveled for each two-second interval. Since speed decreases throughout, we know that the left-hand sum will be an overestimate to the distance traveled, and the right-hand sum an underestimate. Applying the formulas for these sums with $\Delta t = 2$ gives:

$$\text{LEFT} = 2(100 + 80 + 50 + 25 + 10) = 530 \text{ ft}$$
$$\text{RIGHT} = 2(80 + 50 + 25 + 10 + 0) = 330 \text{ ft}$$

(a) The best estimate of the distance traveled will be the average of these two estimates, or $\frac{530+330}{2} =$ 430 ft.

(b) All we can be sure of is that the distance traveled lies between the upper and lower estimates calculated above. In other words, all the black-box data tells us for sure is that the car traveled between 330 and 530 feet before stopping. So we can't be completely sure about whether it hit the skunk or not: answer (ii).

Problem code: PLARZ

14. Roger runs a marathon. His friend Jeff rides behind him on a bicycle and records his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:

Time since Start (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

(a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the **first half hour**.

(b) Give upper and lower estimates for the distance Roger ran in total during the **entire hour and a half**.

(c) How often would Jeff have needed to measure Roger's speed in order to find lower and upper estimates within 0.1 mile of the actual distance he ran?

(a) Note that 15 minutes equals 0.25 hours.
For this part we are only interested in the first two intervals (covering half an hour).
If we always take the highest speed on each interval, we'll get an upper estimate; if we take the lower speed, we'll get a lower estimate. Since Roger's speed is always decreasing, the left-hand sum will always be the upper estimate, and the right-hand sum will be the lower.
Upper (left) estimate $= 12(0.25) + 11(0.25) = 5.75$ miles.
Lower (right) estimate $= 11(0.25) + 10(0.25) =$ 5.25 miles.

(b) In this part, we are interested in all the intervals in the table (full 90 minutes).
Upper (left) estimate $= 12(0.25) + 11(0.25) +$

$10(0.25) + 10(0.25) + 8(0.25) + 7(0.25)$
$= 14.5$ miles.
Lower (right) estimate $= 11(0.25) + 10(0.25) + 10(0.25) + 8(0.25) + 7(0.25) + 0(0.25)$
$= 11.5$ miles.

(c) The difference between Roger's pace at the beginning and the end of his run is 12 mph. If the time between the measurements is Δt, then the difference between the upper and lower estimates is 12 Δt. We want $12\ \Delta t < 0.1$, so

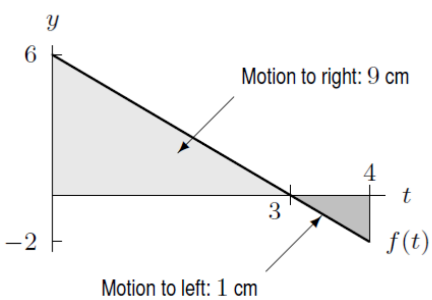
$$\Delta t < \frac{0.1}{12} \approx 0.0083 \text{ hours} = 30 \text{ seconds}$$

Thus Jeff would have to measure Roger's pace every 30 seconds if he wanted the distance estimates to be within 0.1 mile of the actual distance covered.

Problem code: FHFCF

15. The velocity of a particle moving along the x-axis is given by $f(t) = 6 - 2t$ cm/sec. Use a graph of $f(t)$ to find the exact change in position of the particle from time $t = 0$ to $t = 4$ seconds.

The change in position is calculated from the area between the velocity graph and the t-axis, with the region below the axis corresponding to negatives velocities and counting negatively.



The figure above shows the graph of $f(t)$. From $t = 0$ to $t = 3$ the velocity is positive. The region under the graph of $f(t)$ is a triangle with height 6 cm/sec and base 3 seconds. Thus, from $t = 0$ to $t = 3$, the particle moves

$$\text{Distance moved to right } = \frac{1}{2} \cdot 3 \cdot 6 = 9 \text{ centimeters.}$$

From $t = 3$ to $t = 4$, the velocity is negative. The region between the graph of $f(t)$ and the t-axis is a triangle with height 2 cm/sec and base 1 second, so in this interval the particle moves

$$\text{Distance moved to left } = \frac{1}{2} \cdot 1 \cdot 2 = 1 \text{ centimeter.}$$

Thus, the total change in position is 9 - 1 = 8 centimeters to the right.

Problem code: ZHJGY

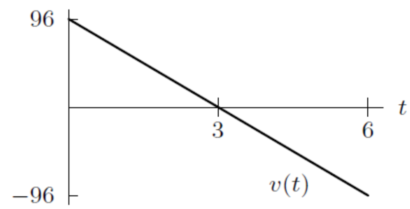
16. A baseball thrown directly upward at 96 ft/sec has velocity $v(t) = 96 - 32t$ ft/ sec at time t seconds.

(a) Graph the velocity from $t = 0$ to $t = 6$.

(b) When does the baseball reach the peak of its flight? How high does it go?

(c) How high is the baseball at time $t = 5$?

(a) See the figure below.



(b) The peak of the flight is when the velocity is 0, namely $t = 3$. The height at $t = 3$ is given by the area under the graph of the velocity from $t = 0$ to $t = 3$; see the figure above. The region is a triangle of base 3 seconds and altitude 96 ft/sec, so the height is $(1/2)3 \cdot 96 = 144$ feet.

(c) The velocity is negative from $t = 3$ to $t = 5$, so the motion is downward then. The distance traveled downward can be calculated by the area of the triangular region which has base of 2 seconds and altitude of -64 ft/sec. Thus, the baseball travels $(1/2)2 \cdot 64 = 64$ feet downward from its peak height of 144 feet at $t = 3$. Thus, the height at time $t = 5$ is the total change in position, $144 - 64 = 80$ feet.

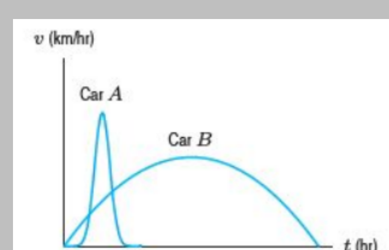
Problem code: NAQHP

17. Two cars start at the same time and travel in the same direction along a straight road. The graph below gives the velocity, v, of each car as a function of time, t. Which car:

(a) Attains the larger maximum velocity?

(b) Stops first?

(c) Travels farther?



(a) Car A has the largest maximum velocity because the peak of car A's velocity curve is higher than the peak of B's.

(b) Car A stops first because the curve representing its velocity hits zero (on the t-axis) first.

(c) Car B travels farther because the **area** under car B's velocity curve is the larger, and area under the velocity graph represents distance.

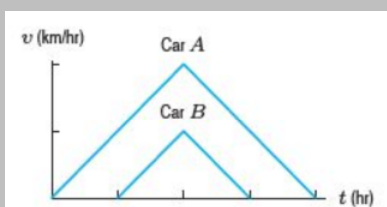
Problem code: XBVMJ

18. Two cars travel in the same direction along a straight road. The graph below shows the velocity, v, of each car at time t. Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/ hr.

    (a) For approximately how long does each car travel?

    (b) Estimate car A's maximum velocity.

    (c) Approximately how far does each car travel?



(a) Since car B starts at $t = 2$, the tick marks on the horizontal axis (which we assume are equally spaced) are 2 hours apart. Thus car B stops at $t = 6$ and travels for 4 hours. Car A starts at $t = 0$ and stops at $t = 8$, so it travels for 8 hours.

(b) Car A's maximum velocity is approximately twice that of car B (50 km/h), so A's max velocity is 100 km/hr.

(c) The distance traveled is given by the area of under the velocity graph. Using the formula for the area of a triangle, the distances are given by

$$\begin{aligned}\text{Car A travels} &= \frac{1}{2} \cdot \text{Base } \cdot \text{Height}\\ &= \frac{1}{2} \cdot 8 \cdot 100 = 400 \text{ km}\\ \text{Car B travels} &= \frac{1}{2} \cdot \text{Base } \cdot \text{Height}\\ &= \frac{1}{2} \cdot 4 \cdot 50 = 100 \text{ km.}\end{aligned}$$

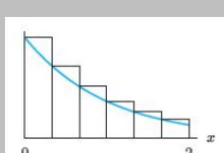
Problem code: GXGBQ

The Definite Integral

19. The figure below shows a Riemann sum approximation with n subdivisions to $\int_a^b f(x)dx$.

    (a) Is it a left- or right-hand approximation? Would the other one be larger or smaller?

    (b) What are a, b, n and Δx?

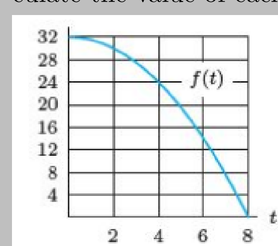


(a) Left-hand sum. Right-hand sum would be smaller.

(b) We have $a = 0$, $b = 2$, $n = 6$, $\Delta x = \frac{2}{6} = \frac{1}{3}$.

Problem code: MBKNB

20. Using the figure below, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \leq t \leq 8$. Calculate the value of each sum.

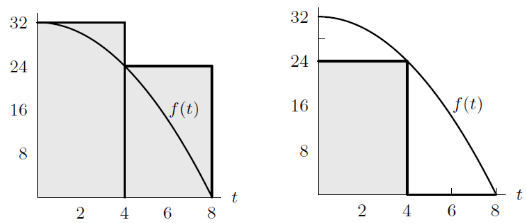


    (a) Left-hand sum with $\Delta t = 4$.

    (b) Right-hand sum with $\Delta t = 4$.

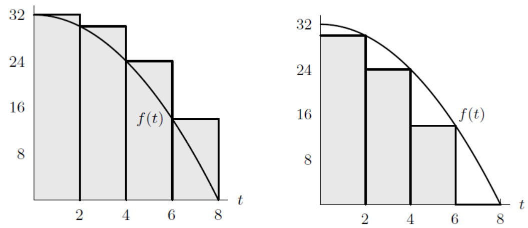
    (c) Left-hand sum with $\Delta t = 2$.

    (d) Right-hand sum with $\Delta t = 2$.



(a) (b)

(a) The left-hand sum with $n = 2$ intervals, or $\Delta t = 4$: $32 \cdot 4 + 24 \cdot 4 = 224$.
(b) The right-hand sum with $n = 2$ intervals, or $\Delta t = 4$: $24 \cdot 4 + 0 \cdot 4 = 96$.

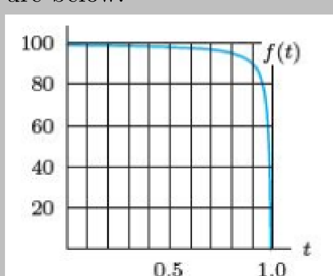


(c) (d)

(c) Left-hand sum with $n = 4$ intervals, or $\Delta t = 2$: $32 \cdot 2 + 30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 = 200$.
(d) Right-hand sum with $n = 4$ intervals, or $\Delta t = 2$: $30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 + 0 \cdot 2 = 136$.

Problem code: EWPAZ

21. The graph of a function $f(t)$ is given in the figure below.



Which of the following four numbers could be an estimate of $\int_0^1 f(t)\, dt$, accurate to two decimal places? Explain how you chose your answer.
(a) -98.35 (b) 71.84 (c) 100.12 (d) 93.47

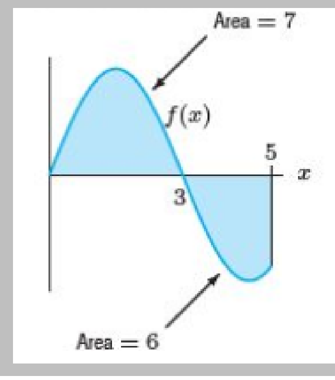
The graph given shows that f is positive for $0 \leq t \leq 1$, so the integral value must be positive: the answer cannot be -98.35.

Since the graph is contained within a rectangle of height 100 and length 1, the answer 100.12 is too large.

The graph of f is well above the horizontal line y = 80 for $0 \leq t \leq 0.95$, so the integral is likely much higher then $71.84 < 80$, so out of the choices given the best estimate is 93.47: answer (d).

Problem code: UTKXU

22. (a) What is the area between the graph of $f(x)$ shown below and the x-axis, between $x = 0$ and $x = 5$?
(b) What is $\int_0^5 f(x)\, dx$?



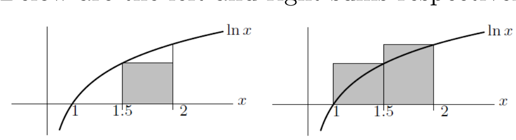
(a) The total area between $f(x)$ and the x-axis is the sum of the two given areas, so area $= 7 + 6 = 13$.
(b) To find the integral, we note that from $x = 3$ to $x = 5$, the function lies below the x-axis, and hence makes a negative contribution to the integral. So

$$\int_0^5 f(x)\, dx = \int_0^3 f(x)\, dx + \int_3^5 f(x) dx = 7 - 6 = 1$$

Problem code: CTYZR (Video Solution by K.C.)

23. (a) On a sketch of $y = \ln(x)$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\, dx$. Write out the terms in the sum, but do not evaluate it.
(b) On another sketch, represent the *right* Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\, dx$. Write out the terms in the sum, but do not evaluate it.
(c) Which sum is an overestimate? Which sum is an underestimate?

(a) Below are the left and right sums respectively.



$$\begin{aligned} \text{Left sum} &= f(1)\Delta x + f(1.5)\Delta x \\ &= \underbrace{(\ln 1)}_{=0} 0.5 + \ln(1.5)0.5 = (\ln 1.5)0.5 \end{aligned}$$

(b)

$$\begin{aligned} \text{Right sum} &= f(1.5)\Delta x + f(2)\Delta x \\ &= (\ln 1.5)0.5 + (\ln 2)0.5 \end{aligned}$$

(c) Right sum is an overestimate, left sum is an underestimate.

Problem code: HGALL

24. Estimate $\int_1^2 x^2\ dx$ using left- and right-hand sums with four subdivisions, and then averaging them. How far from the true value of the integral could your final estimate be?

Left-hand sum gives:
$1^2(1/4) + (1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) = 1.96875$.

Right-hand sum gives:
$(1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) + (2)^2(1/4) = 2.71875$.

We improve our estimate the value of the integral by taking the average of these two sums, which is 2.34375.

Since x^2 is always increasing on $1 \le x \le 2$, the true value of the integral lies between 1.96875 and 2.71875. Thus the most our estimate could be off is 0.375 (half of the range from 2.34 to the lower and upper bounds 1.97 and 2.72). We expect our 2.34 estimate to be much closer than that possible error bound though. (And it is: the true value of the integral is $7/3 \approx 2.333$.)

Problem code: YTXSZ

25. Without computing the sums, find the difference between the right- and left-hand Riemann sums if we use $n = 500$ subintervals to approximate $\int_{-1}^{1} (2x^3 + 4)\ dx$.

We have $\Delta x = 2/500 = 1/250$. The formulas for the left- and right-hand Riemann sums give us that

$$\begin{aligned} \text{Left} &= \Delta x[f(-1) + f(-1+\Delta x) + \ldots \\ &\quad + f(1-2\Delta x) + f(1-\Delta x)] \\ \text{Right} &= \Delta x[f(-1+\Delta x) + f(-1+2\Delta x) + \ldots \\ &\quad + f(1-\Delta x) + f(1)] : \end{aligned}$$

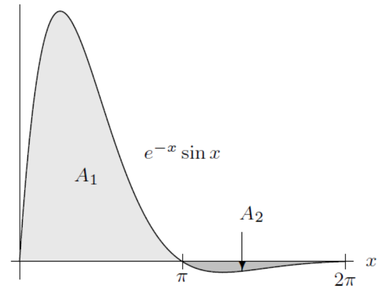
Subtracting these yields

$$\begin{aligned} \text{Right - Left} &= \Delta x[f(1) - f(-1)] \\ &= \frac{1}{250}[6-2] = \frac{4}{250} = \frac{2}{125}. \end{aligned}$$

The estimates from the Right and Left sums will differ by 2/125.

Problem code: DLAXY

26. Without computation, decide if $\int_0^{2\pi} e^{-x} \sin x\ dx$ is positive or negative. [Hint: Sketch $e^{-x}\sin(x)$].



Looking at the graph of $y = e^{-x}\sin x$ shown above (remember the sketching of variable amplitudes earlier in the course!), for $0 \le x \le 2\pi$, we see that the area, A_1, which contributes a positive amount to the integral $\int_0^{2\pi} e^{-x}\sin(x)\ dx$, is much larger than the area A_2, which contributes a negative amount to the integral.

Since the positive integral contributions are larger than the negative, the overall integral will be positive.

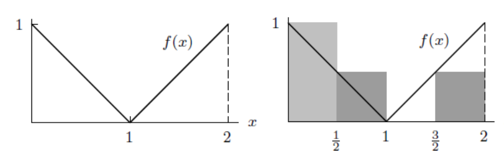
Problem code: NPYDY (Video Solution by K.C.)

27. (a) Graph $f(x) = \begin{cases} 1-x & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 1 < x \le 2 \end{cases}$

(b) Find the *exact* value of $\int_0^2 f(x)\ dx$ (hint: sketch and see what shapes you get).

(c) Calculate the 4-term left Riemann sum approximation to the definite integral. How does the approximation compare to the exact value?

(a)



(b) The area is made up of two triangles, with total area of $2 \times (1/2)(1)(1) = 1$ square unit.

(c) Using $\Delta x = 1/2$ in the 4-term Riemann sum shown in right-side graph above, we have

$$\begin{aligned} &\text{Left hand sum} \\ &= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \\ &= 1\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) \\ &= 1. \end{aligned}$$

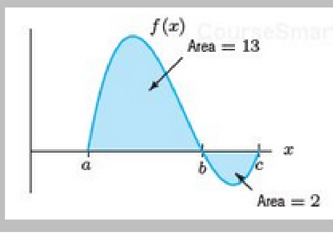
We notice that in this case the approximation is exactly equal to the exact value of the integral. This is mostly coincidence due to the simple shape of $f(x)$. In general, approximations will not work out to be exactly the same as the value of the integral.

Problem code: MUWAF

28. Using the figure below, find the values of

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$

(c) $\int_a^c f(x)\,dx$ (d) $\int_a^c |f(x)|\,dx$

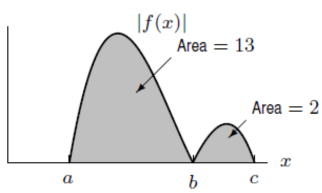


(a) The area between the graph of $f(x)$ and the x-axis between $x = a$ and $x = b$ is 13, so $\int_a^b f(x)\,dx = 13$.

(b) Since the graph of $f(x)$ is below the x-axis for $b < x < c$, $\int_b^c f(x)\,dx = -2$.

(c) Since the graph of $f(x)$ is above the x-axis for $a < x < b$ and below for $b < x < c$, $\int_a^c f(x)\,dx = 13 - 2 = 11$.

(d) The graph of $|f(x)|$ is the same as the graph of $f(x)$, except that the part *below* the x-axis is reflected to be *above* it (see graph of $|f(x)|$ below). Thus $\int_a^c |f(x)|\,dx = 13 + 2 = 15$.

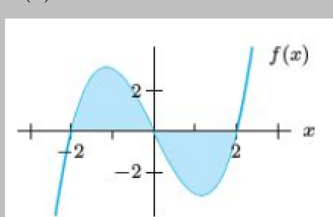


Problem code: DCCTF

29. Given the figure below, and the statement that $\int_{-2}^0 f(x)\,dx = 4$, estimate

(a) $\int_0^2 f(x)\,dx$ (b) $\int_{-2}^2 f(x)\,dx$

(c) The total shaded area.



The region shaded between $x = 0$ and $x = 2$ appears to have approximately the same area as the region shaded between $x = -2$ and $x = 0$, but it lies below the axis. Since $\int_{-2}^0 f(x)\,dx = 4$, we have the following results:

(a) $\int_0^2 f(x)\,dx \approx$ negative of $\int_{-2}^0 f(x)\,dx = -4$

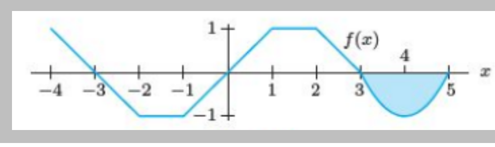
(b) $\int_{-2}^2 f(x)\,dx \approx 4 - 4 = 0$.

(c) The total area shaded is approximately $4 + 4 = 8$.

Problem code: ZCUDM

30. (a) Using the graph below, find $\int_{-3}^0 f(x)\,dx$.

(b) If the area of the shaded region is A, estimate $\int_{-3}^4 f(x)\,dx$.



(a) $\int_{-3}^0 f(x)\,dx = -2$: counting squares or computing areas of the triangles and rectangles in this region.

(b) We break the integral over the interval $x = -3 \ldots 5$ into pieces, each of which we can find the areas of.

$$\begin{aligned}\int_{-3}^4 f(x)\,dx &= \int_{-3}^0 f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx \\ &= -2 + 2 - \frac{A}{2} \\ &= \frac{-A}{2}\end{aligned}$$

Problem code: RHKVQ

31. Calculate the following approximations to $\int_0^6 x^2\,dx$. (a) LEFT(2); (b) RIGHT(2); (c) TRAP(2); (d) MID(2)

(a) Since two rectangles are being used, the width of each rectangle is $\Delta x = (6 - 0)/2 = 3$. The height is given by the left-hand endpoint so we have

$$\begin{aligned}\text{LEFT}(2) &= f(0)\cdot 3 + f(3)\cdot 3 \\ &= 0^2\cdot 3 + 3^2\cdot 3 = 27\end{aligned}$$

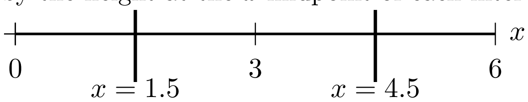
(b) Again, $\Delta x = 3$. The height of each rectangle is given by the right-hand endpoint so we have

$$\begin{aligned}\text{RIGHT}(2) &= f(3)\cdot 3 + f(6)\cdot 3 \\ &= 9\cdot 3 + 36\cdot 3 = 135\end{aligned}$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$TRAP(2) = \frac{27+135}{2} = 81$$

(d) $\Delta x = 3$ again. The height of each rectangle is given by the height at the x midpoint of each interval:



so we have

$$\begin{aligned} MID(2) &= \sum f(x)\ \Delta x \\ &= f(1.5)\cdot 3 + f(4.5)\cdot 3 \\ &= (1.5)^2\cdot 3 + (4.5)^2 \cdot 3 \\ &= 67.5 \end{aligned}$$

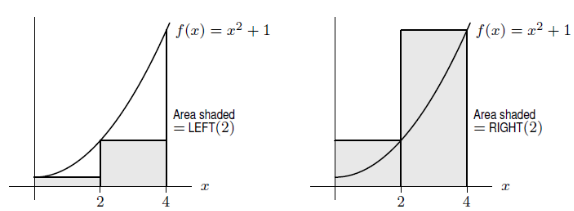
Problem code: LTVAG

32. (a) Find LEFT(2) and RIGHT(2) for $\int_0^4 (x^2 + 1)\ dx$.
    (b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

(a)

$$\begin{aligned} \text{LEFT}(2) &= 2\cdot f(0) + 2\cdot f(2) \\ &= 2\cdot 1 + 2\cdot 5 \\ &= 12 \\ \text{RIGHT}(2) &= 2\cdot f(2) + 2\cdot f(4) \\ &= 2\cdot 5 + 2\cdot 17 \\ &= 44 \end{aligned}$$

(b)



LEFT(2) will be an under-estimate, while RIGHT(2) will be an over-estimate.

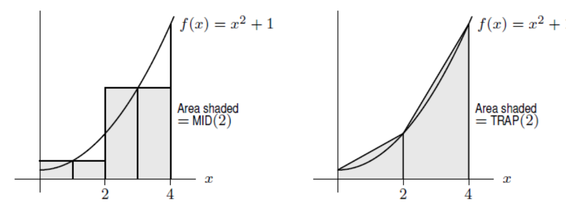
Problem code: TZBQQ

33. (a) Find MID(2) and TRAP(2) for $\int_0^4 (x^2 + 1)\ dx$.
    (b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

(a)

$$\begin{aligned} \text{MID}(2) &= 2\cdot f(1) + 2\cdot f(3) \\ &= 2\cdot 2 + 2\cdot 10 \\ &= 24 \\ \text{TRAP}(2) &= \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} \\ &= \frac{12+44}{2} \\ &= 28 \end{aligned}$$

(b)



MID(2) is an underestimate, since $f(x) = x^2 + 1$ is concave up; this means that the new growth of the function after the midpoint (missed by MID) will be greater than extra area before (captured by MID).

Since $f(x)$ is concave up, TRAP(2) is an overestimate, because the trapezoidal secant lines in the TRAP rule will always lie above the original curve, leading to more area in the trapezoids than under the curve.

Problem code: XRLNB

34. Calculate the following approximations to $\int_0^\pi \sin(\theta)d\theta$. (a) LEFT(2); (b) RIGHT(2); (c) TRAP(2); (d) MID(2)

(a) Since two rectangles are being used, the width of each rectangle is $\Delta x = (\pi - 0)/2 = \pi/2$. The height is given by the left-hand endpoint so we have

$$\begin{aligned} \text{LEFT}(2) &= f(0)\cdot\frac{\pi}{2} + f(\pi/2)\cdot\frac{\pi}{2} \\ &= \sin 0\cdot\frac{\pi}{2} + \sin(\pi/2)\cdot\frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

(b) Again $\Delta x = \pi/2$. The height of each rectangle is given by the right-hand endpoint so we have

$$\begin{aligned} RIGHT(2) &= f(\pi/2)\cdot\frac{\pi}{2} + f(\pi)\cdot\frac{\pi}{2} \\ &= \sin(\pi/2)\cdot\frac{\pi}{2} + \sin(\pi)\cdot\frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$\text{TRAP}(2) = \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} = \frac{\pi/2 + \pi/2}{2} = \frac{\pi}{2}$$

(d) Again, $\Delta x = \pi/2$. The height of each rectangle is given by the height at the midpoint so we have

$$\begin{aligned} \text{MID}(2) &= f(\pi/4) \cdot \frac{\pi}{2} + f(3\pi/4) \cdot \frac{\pi}{2} \\ &= \sin(\pi/4) \cdot \frac{\pi}{2} + \sin(3\pi/4) \cdot \frac{\pi}{2} \\ &= \frac{\sqrt{2}\pi}{2} \end{aligned}$$

Problem code: PWUCS

35. Using the table below, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using LEFT, RIGHT, and TRAP.

Time, t (s)	0	1	2	3	4	5	6
Velocity, v (m/s)	3	4	5	4	7	8	11

Let $s(t)$ be the distance traveled at time t, and $v(t)$ be the velocity at time t. Then the distance traveled during the interval $0 \le t \le 6$ is given by $\int_0^6 v(t)\, dt$.

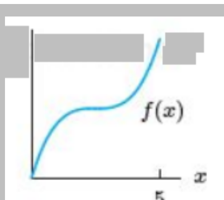
We estimate the distance by estimating this integral. From the table, we find:

- LEFT(6) = (Δt)(sum of 6 LEFT velocity values) $= (1)(3 + 4 + 5 + 4 + 7 + 8) = 31$ m;
- RIGHT(6) = (Δt)(sum of 6 RIGHT velocity values) $= (1)(4 + 5 + 4 + 7 + 8 + 11) = 39$ m;
- TRAP(6) = longer calculation using the (average velocity time Δt) on each interval, **or** average of LEFT(6) and RIGHT(6) = $\frac{31 + 39}{2} = 35$ m.

Problem code: UYWPM

For the functions in Problems 36 – 39, pick which approximation- left, right, trapezoid, or midpoint- is guaranteed to give an *overestimate* for $\int_0^5 f(x)\, dx$, and which is guaranteed to give an *underestimate*. (There may be more than one.)

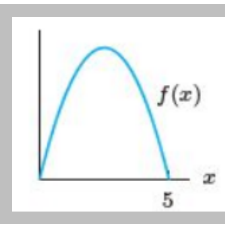
36.



$f(x)$ is increasing, so RIGHT gives an overestimate and LEFT gives an underestimate.

Problem code: EXUCW

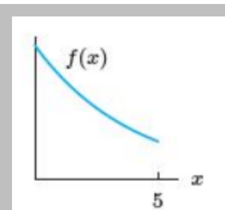
37.



$f(x)$ is concave down, so MID gives an overestimate and TRAP gives an underestimate.

Problem code: WCZPR

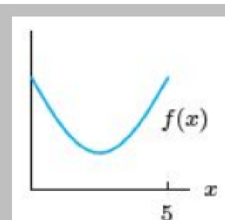
38.



$f(x)$ is decreasing and concave up, so LEFT and TRAP give overestimates and RIGHT and MID give underestimates.

Problem code: QCUZQ

39.



$f(x)$ is concave up, so TRAP gives an overestimate and MID gives an underestimate.

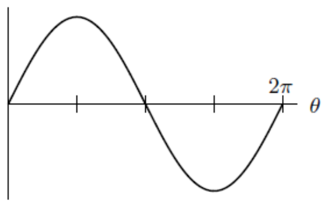
Problem code: DTGNX

40. (a) Find the exact value of $\int_0^{2\pi} \sin\theta\, d\theta$ without calculation (i.e. from a sketch).

(b) Explain, using pictures, why the MID(1) and MID(2) approximations to this integral give the exact value.

(c) Does MID(3) give the exact value of this integral? How about MID(n)? Explain.

(a)



From the graph of sine above, and the fact that the integral includes a complete $[0, 2\pi]$ cycle, it is clear from symmetry that the positive and negative contributions to the integral must be equal, so

$$\int_0^{2\pi} \sin(\theta)\, d\theta = 0$$

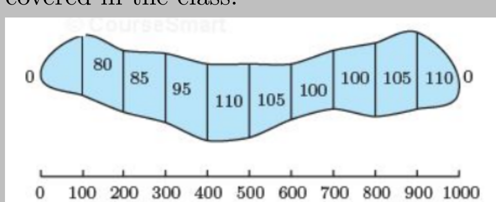
(b) Again, refer to the graph above. MID(1) is 0 since the midpoint of 0 and 2π is π and $\sin\pi = 0$. Thus MID(1) $= 2\pi \sin\pi = 0$.

The midpoints we use for MID(2) are $\pi/2$ and $3\pi/2$, and $\sin(\pi/2) = -\sin(3\pi/2)$. Thus MID(2) $= \pi\sin(\pi/2) + \pi\sin(3\pi/2) = 0$.

(c) MID(3) $= 0$. In general, MID(n) $= 0$ for all n, even though your calculator (because of round-off error) might not return it as such. The reason is that $\sin(x) = -\sin(2\pi - x)$. If we use MID($n$), we will always take sums where we are adding pairs of the form $\sin(x)$ and $\sin(2\pi - x)$, so the sum will cancel to 0. (If n is odd, we will get a $\sin\pi$ in the sum which does not pair up with anything, but $\sin\pi$ is already 0.)

Problem code: GQZHW

41. The width, in feet, at various points along the fairway of a hole on a golf course is given in the figure below. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway. Select the most accurate estimate approach from the methods covered in the class.



Of our approaches, the two with higher accuracy are TRAP and MID. However, we have information in the diagram about the width of the golf course fairway at the ends of each interval, which is the information we would need for a TRAP estimate.

We approximate the area of the playing field by using Riemann sums. From the data provided, we can first estimate the area of the fairway:

$$\begin{aligned} &TRAP(10) \\ &= \left(\frac{0+80}{2}\right)(100) + \left(\frac{80+85}{2}\right)(100) + \dots \\ &\quad + \left(\frac{105+110}{2}\right)(100) + \left(\frac{110+0}{2}\right)(100) \\ &= 89{,}000 \text{ square feet.} \end{aligned}$$

With that area, and the fact that one pound of fertilizer covers 200 square feet, approximately $\dfrac{89{,}000 \text{ sq. ft.}}{200\text{sq. ft./lb}} =$ 445 lbs. of fertilizer should be necessary.

Note: for this problem, if you had used LEFT or RIGHT to estimate the fairway area, you would find that all 3 estimates are the same. The fact that LEFT=RIGHT=TRAP here doesn't mean there is no error in the estimate; it actually means that the error is completely unknown. You will always have this relationship when the first and last heights are equal because that makes LEFT = RIGHT, and then TRAP (= average of LEFT and RIGHT) will equal the same value as well.

Problem code: WDSDD