Week #5 - Implicit Derivatives, Related Rates and Optimization

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Implicit Differentiation

1. Consider the graph implied by the equation $xy^2 = 1$.

   What is the equation of the line through $(\frac{1}{4}, 2)$ which is also tangent to the graph?

2. Consider the circle defined by $x^2 + y^2 = 25$

   Find the equations of the tangent lines to the circle where $x = 4$.

3. Calculate the derivative of y with respect to x, given that
$$x^4 y + 4xy^4 = x + y$$

4. Calculate the derivative of y with respect to x, given that
$$xe^y = 4xy + 5y^4$$

5. Use implicit differentiation to find the equation of the tangent line to the curve $xy^3 + xy = 14$ at the point $(7, 1)$.

6. Find dy/dx by implicit differentiation.
$$\sqrt{x+y} = 9 + x^2y^2$$

7. Find all the x-coordinates of the points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

8. The curve with equation $2y^3+y^2-y^5 = x^4-2x^3+x^2$ has been likened to a bouncing wagon (graph it to see why). Find the x-coordinates of the points on this curve that have horizontal tangents.

9. Use implicit differentiation to find the (x, y) points where the **circle** defined by
$$x^2 + y^2 - 2x - 4y = -1$$
   has horizontal and vertical tangent lines.

   (a) Find the points where the curve has a horizontal tangent line.

   (b) Find the points where the curve has a vertical tangent line.

10. The relation
$$x^2 - 2xy + y^2 + 6x - 10y + 29 = 0$$
    defines a parabola.

    (a) Find the points where the curve has a horizontal tangent line.

    (b) Find the points where the curve has a vertical tangent line.

Related Rates

11. Gravel is being dumped from a conveyor belt at a rate of 30 cubic feet per minute. It forms a pile in the shape of a right circular cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 17 feet high?
    Recall that the volume of a right circular cone with height h and radius of the base r is given by $V = \frac{1}{3}\pi r^2 h$.

12. When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$ where C is a constant. Suppose that at a certain instant the volume is 550 cubic centimeters, and the pressure is 91 kPa and is decreasing at a rate of 7 kPa/minute. At what rate in cubic centimeters per minute is the volume increasing at this instant?

    (Pa stands for Pascal – it is equivalent to one Newton/(meter squared); kPa is a kiloPascal or 1000 Pascals.)

13. At noon, ship A is 40 nautical miles due west of ship B. Ship A is sailing west at 23 knots and ship B is sailing north at 23 knots. How fast (in knots) is the distance between the ships changing at 6 PM? (Note: 1 knot is a speed of 1 nautical mile per hour.)

14. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 3 cm^2/min.

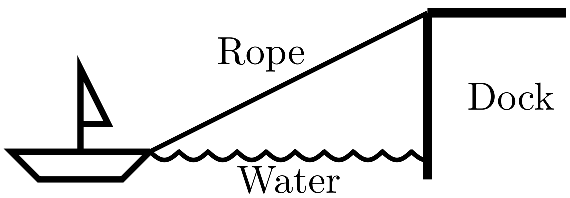
    At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 105 cm^2 ?

15. A street light is at the top of a 11 ft tall pole. A woman 6 ft tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of

her shadow moving when she is 45 ft from the base of the pole?

16. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat.

    If the rope is pulled in at a rate of 1.2 m/s, how fast is the boat approaching the dock when it is 9 m from the dock?



17. Water is leaking out of an inverted conical tank at a rate of 12000.0 cubic centimeters per min at the same time that water is being pumped *into* the tank at a constant rate.

    - The tank has height 8.0 meters and the diameter at the top is 5.0 meters.
    - The depth of the water is increasing at 28.0 centimeters per minute when the height of the water is 4.0 meters.

    Find the rate at which water is being pumped into the tank, in cubic centimeters per minute.

18. A spherical snowball is melting in such a way that its diameter is decreasing at rate of 0.4 cm/min. At what rate is the volume of the snowball decreasing when the diameter is 17 cm? (Note the answer is a positive number).

19. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V is $PV = nRT$, where n is the number of moles of the gas and $R = .0821$ is the gas constant. Suppose that, at a certain instant, $P = 8$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ moles.

20. A potter forms a piece of clay into a right circular cylinder. As she rolls it, the height h of the cylinder increases and the radius r decreases. Assume that no clay is lost in the process. Suppose the height of the cylinder is increasing by 0.4 centimeters per second. What is the rate at which the radius is changing when the radius is 3 centimeters and the height is 12 centimeters?

21. A hot air balloon rising vertically is tracked by an observer located 2 miles from the lift-off point. At a certain moment, the angle between the observer's line-of-sight and the horizontal is $\frac{\pi}{6}$, and it is changing at a rate of 0.1 rad/min. How fast is the balloon rising at this moment?

22. A road perpendicular to a highway leads to a farmhouse located 5 mile away. A car traveling on the highway passes through this intersection at a speed of 55mph.

    How fast is the distance between the car and the farmhouse increasing when the car is 7 miles past the intersection of the highway and the road?

23. Assume that the radius r of a sphere is expanding at a rate of 6in./min. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

    Determine the rate at which the volume is changing with respect to time when $r = 11$in.

24. The radius of a circular oil slick expands at a rate of 7 m/min.

    (a) How fast is the area of the oil slick increasing when the radius is 26 m?

    (b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 2 mins?

25. A searchlight rotates at a rate of 2 revolutions per minute. The beam hits a wall located 10 miles away and produces a dot of light that moves horizontally along the wall. How fast (in miles per hour) is this dot moving when the angle θ between the beam and the line through the searchlight perpendicular to the wall is $\frac{\pi}{6}$? Note that $d\theta/dt = 2(2\pi) = 4\pi$ rad/minute.

26. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

Optimization Introduction

27. Let $f(x) = x^2 - 10x + 13$, and consider the interval $[0, 10]$.

    (a) Find the critical point c of $f(x)$ and compute $f(c)$.

    (b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 10]$.

    (c) Determine the global min and max of $f(x)$ on $[0, 10]$.

    (d) Find the global min and max of $f(x)$ on $[0, 1]$. (Note: not the same interval as before)

28. Find the maximum and minimum values of the func-

tion $f(x) = x - \dfrac{125x}{x+5}$ on the interval [0,21].

29. The function $f(x) = -2x^3 + 21x^2 - 36x + 10$ has one local minimum and one local maximum. Find their (x, y) locations.

30. A Queen's University student decided to depart from Earth after his graduation to find work on Mars. Before building a shuttle, he conducted careful calculations. A model for the velocity of the shuttle, from liftoff at t = 0 s until the solid rocket boosters were jettisoned at t = 80 s, is given by

$$v(t) = 0.001094333t^3 - 0.08215t^2 + 28.6t - 4.3$$

(in feet per second). Using this model, estimate the global maximum value and global minimum value of the **acceleration** of the shuttle between liftoff and the jettisoning of the boosters.

31. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where μ is a positive constant called the coefficient of friction and where $0 \le \theta \le \pi/2$. Find the value for $\tan\theta$ which minimizes the force. Your answer may depend on W and μ.

32. A ball is thrown up on the surface of a moon. Its height above the lunar surface (in feet) after t seconds is given by the formula

$$h = 217t - \frac{7}{4}t^2.$$

(a) Find the time that the ball reaches its maximum height.

(b) Find the maximal height attained by the ball.

Optimization Word Problems

33. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. Suppose that one has a rule that the sum of the length, width and height of any piece of luggage must be less than or equal to 192 cm. A passenger wants to take a box of the maximum allowable volume.

(a) If the length and width are to be equal, what should the dimensions be?

(b) In this case, what is the volume?

(c) If the length is be twice the width, what should the dimensions be?

(d) In this case, what is the volume?

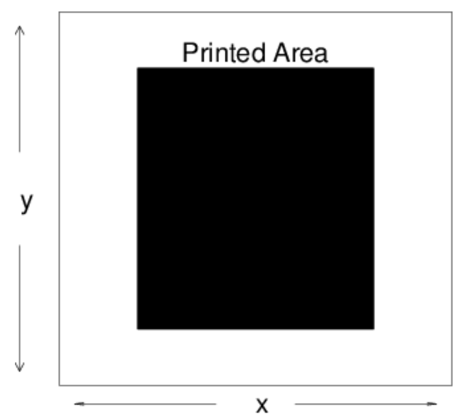
Include units in all your answers.

34. A wire 3 meters long is cut into two pieces. One piece is bent into a square for a frame for a stained glass ornament, while the other piece is bent into a circle for a TV antenna.

(a) To reduce storage space, where should the wire be cut to **minimize** the total area of both figures?

(b) Where should the wire be cut to **maximize** the total area?

35. A printed poster is to have a total area of 799 square inches with top and bottom margins of 6 inches and side margins of 4 inches. What should be the dimensions of the poster so that the printed area be as large as possible? Let x denote the width of the poster and let y denote the length.



(a) Write the function of x and y that you need to maximize.

(b) Express that function in terms of x alone.

(c) Find the critical points of the function.

(d) Use the second derivative test to verify that $f(x)$ has a maximum at this critical point

(e) Find the optimal dimensions of the poster, and the resulting area. Include units.

36. A box with an open top has vertical sides, a square bottom, and a volume of 32 cubic meters. If the box has the least possible surface area, find its dimensions.

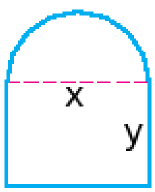
37. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle with sides of length 2 if one side of the rectangle lies on the base of the triangle.

38. Find the minimum distance from the parabola

$$x - y^2 = 0$$

to the point (0,3).

39. Suppose that 241 ft of fencing are used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as the following figure:



Find the dimensions of the corral with maximum area.

40. Find the maximum area of a triangle formed in the first quadrant by the x-axis, y-axis and a tangent line to the graph of $f = (x+2)^{-2}$.

41. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs $4 per square foot and the metal for the sides costs $6 per square foot. Find the dimensions that minimize cost if the box has a volume of 35 cubic feet.

42. A rectangle is inscribed with its base on the x axis and its upper corners on the parabola $y = 12 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

43. Centerville is the headquarters of Greedy Cablevision Inc. The cable company is about to expand service to two nearby towns, Springfield and Shelbyville. There needs to be cable connecting Centerville to both towns. The idea is to save on the cost of cable by arranging the cable in a Y-shaped configuration. Centerville is located at $(8, 0)$ in the xy-plane, Springfield is at $(0, 5)$, and Shelbyville is at $(0, -5)$. The cable runs from Centerville to some point $(x, 0)$ on the x-axis where it splits into two branches going to Springfield and Shelbyville. Find the location $(x, 0)$ that will minimize the amount of cable between the 3 towns and compute the amount of cable needed. Justify your answer.

(a) What function of x needs to be minimized to solve this problem?

(b) Find the critical points of $f(x)$.

(c) Use the second derivative test to verify that $f(x)$ has a minimum at this critical point.

(d) Compute the minimum amount of wire needed.

44. A cylinder is inscribed in a right circular cone of height 4 and radius (at the base) equal to 3.5. What are the dimensions of such a cylinder which has maximum volume?

45. The Nearsighted Cow Problem: A Calculus Classic.

A rectangular billboard 7 feet in height stands in a field so that its bottom is 13 feet above the ground. A nearsighted cow with eye level at 4 feet above the ground stands x feet from the billboard. Express θ, the vertical angle subtended by the billboard at her eye, in terms of x. Then find the distance x_0 the cow must stand from the billboard to maximize $\theta(x)$.