

Week #4 - Inverse Trig, Tangent Lines and Linearization

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Projectile Motion

1. A projectile is fired from ground level on a flat region with an initial speed of 200 m/s and angle of elevation 60° . Find:
 - (a) The range of the projectile.
 - (b) The maximum height reached.
 - (c) The speed at impact.

Note: this solution is long because it is written in great detail, because the approach is likely new to most students. In practice it would be written out by hand in about 8-12 lines once the general approach was understood.

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 $v_{\text{vec}}(0)|| = 200$ m/s and, since the angle of elevation is 60° , we can break the initial velocity into components as follows:
 $\vec{v}(0) = 200\langle \cos 60^\circ, \sin 60^\circ \rangle = 200\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ m/s or, multiplying in the 200, $\vec{v}(0) = \langle 100, 100\sqrt{3} \rangle$ m/s.

For the initial position, we set our launch location to the origin, $\vec{r}(0) = \langle 0, 0 \rangle$.

That is all the information we have available about the *initial* conditions: now we move to our prediction of the motion *over time*.

Ignoring air resistance, the only force on the projectile is that due to gravity. That force is purely in the y component, at $-mg$, so $\vec{F}(t) = \langle 0, -mg \rangle$ where $g \approx 9.8\text{m/s}^2$.

Using $F = ma$, we can solve for $\vec{a} = \vec{F}/m$, or

$$\vec{a}(t) = (1/m)\langle 0, -mg \rangle = \langle 0, -g \rangle.$$

Once we know the acceleration vector, and it is relatively simple (zeros and constants), we can work backwards to infer what the velocity over time must be.

- In the x component, to get an acceleration of $a_x = 0$, we need v_x to be a constant, say c_1 .
- In the y component, to get an acceleration of $a_y = -g$, we need v_y to be linear: $-gt + c_2$, where c_2 is another possible constant.

This observation gives us the general form of our velocity vector over time:

$$\vec{v}(t) = \langle c_1, -gt + c_2 \rangle.$$

How do we find the value of the constants? Well, we know the velocity at time $t = 0$ from the information in the question. Subbing in $t = 0$ into our velocity function, and setting it equal to the know $\vec{v}(0)$ we get

$$\begin{aligned} \text{at } t = 0, \quad \langle c_1, -g(0) + c_2 \rangle &= \langle 100, 100\sqrt{3} \rangle \\ \langle c_1, c_2 \rangle &= \langle 100, 100\sqrt{3} \rangle \\ \text{or specifically } c_1 &= 100 \text{ and } c_2 = 100\sqrt{3}. \end{aligned}$$

We now have completely defined our velocity function over time: $\vec{v}(t) = \langle 100, -gt + 100\sqrt{3} \rangle$.

We can now work backwards again to infer what the *position* $\vec{r}(t)$ over time must be.

- In the x component, to get a velocity of $v_x = 100$, we need r_x to be a linear function, say $100t + c_3$.
- In the y component, to get a velocity of $v_y = -gt + 100\sqrt{3}$, we need r_y to be a quadratic with the following form: $-\frac{gt^2}{2} + 100\sqrt{3}t + c_4$. You can check by differentiating that this will equal our v_y .

Thus our position over time's formula must be of the form

$$\vec{r}(t) = \langle 100t + c_3, -\frac{gt^2}{2} + 100\sqrt{3}t + c_4 \rangle.$$

We can repeat our use of the initial conditions at $t = 0$ for the position now to find c_3 and c_4 . This time, because our initial position was simply $\langle 0, 0 \rangle$, we find both $c_3 = 0$ and $c_4 = 0$.

So finally, our position over time will be given by $\vec{r}(t) = \langle 100t, \frac{-gt^2}{2} + 100\sqrt{3}t \rangle$ or using $g = 9.8 \text{ m/s}^2$, $\vec{r}(t) = \langle 100t, -4.9t^2 + 100\sqrt{3}t \rangle$.

(a) Parametric equations for the projectile are $x(t) = 100t, y(t) = 100\sqrt{3}t - 4.9t^2$.

The projectile reaches the ground

$$\text{when } y(t) = 0 \text{ (and } t > 0) \Rightarrow 100\sqrt{3}t - 4.9t^2 = t(100\sqrt{3} - 4.9t) = 0 \Rightarrow t = \frac{100\sqrt{3}}{4.9} \approx 35.3 \text{ s.}$$

$$\text{So the range is } x\left(\frac{100\sqrt{3}}{4.9}\right) = 100\left(\frac{100\sqrt{3}}{4.9}\right) \approx 3535 \text{ m.}$$

(b) The maximum height is reached when the vertical component of velocity is 0 (not moving upwards nor downwards):

$$y'(t) = 0 \Rightarrow 100\sqrt{3} - 9.8t = 0 \Rightarrow t = \frac{100\sqrt{3}}{9.8} \approx 17.7 \text{ s.}$$

$$\text{Thus the maximum height is } y\left(\frac{100\sqrt{3}}{9.8}\right) = 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1531 \text{ m.}$$

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3}}{4.9}$ s. Thus, the velocity at impact is

$$\vec{v}\left(\frac{100\sqrt{3}}{4.9}\right) = \langle 100, 100\sqrt{3} - 9.8\left(\frac{100\sqrt{3}}{4.9}\right) \rangle = \langle 100, -100\sqrt{3} \rangle \text{ and the speed is}$$

$$\left| \vec{v}\left(\frac{100\sqrt{3}}{4.9}\right) \right| = \sqrt{(100)^2 + (-100\sqrt{3})^2} = 200 \text{ m/s.}$$

Problem code: ATBTS (Video Solution Part 1 by K.M.) (Video Solution Part 2 by K.M.)

2. Rework question 1 if the projectile is fired from a position 100 m above the ground.

All the work from the previous problem is the same, except that the initial *position* has moved:

- Constant acceleration is still $ds\vec{a}(t) = \langle 0, -g \rangle \text{ m/s}^2$.
- Initial velocity is still $\vec{v}(0) = \langle 100, 100\sqrt{3} \rangle \text{ m/s}$.
- Initial position has changed to $\vec{r}(0) = \langle 0, 100 \rangle \text{ m}$.

The velocity calculations will be the same, because the acceleration and initial velocity are the same the previous problem.

$$\vec{v}(t) = \langle 100, -gt + 100\sqrt{3} \rangle.$$

The difference arises when we use the new initial position vector, $\vec{r}(0) \langle 0, 100 \rangle$. This affects the constants we got, specifically c_4 . Our position function becomes:

$$\vec{r}(t) = \langle 100t, -4.9t^2 + 100\sqrt{3}t + \underbrace{100}_{\text{new}} \rangle.$$

(a) $y = 0 \Rightarrow 100 + 100\sqrt{3}t - 4.9t^2 = 0$ or $4.9t^2 - 100\sqrt{3}t - 100 = 0$. From the quadratic formula we have

$$t = \frac{100\sqrt{3} \pm \sqrt{(-100\sqrt{3})^2 - 4(4.9)(-100)}}{2(4.9)} = \frac{100\sqrt{3} \pm \sqrt{31,960}}{9.8}. \text{ Taking the positive } t\text{-value gives}$$

$$t = \frac{100\sqrt{3} + \sqrt{31,960}}{9.8} \approx 35.9 \text{ s. Thus the range is } x = 100 \cdot \frac{100\sqrt{3} + \sqrt{31,960}}{9.8} \approx 3592 \text{ m.}$$

(b) The maximum height is attained when $\frac{dy}{dt} = 0 \Rightarrow 100\sqrt{3} - 9.8t = 0 \Rightarrow t = \frac{100\sqrt{3}}{9.8} \approx 17.7 \text{ s}$

$$\text{and the maximum height is } 100 + 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1631 \text{ m.}$$

Alternate solution: Because the projectile is fired in the same direction and with the same velocity as in question 2, but from a point 100 m higher, the maximum height reached is 100 m higher than that found in question 2, that is, $1531 \text{ m} + 100 \text{ m} = 1631 \text{ m}$.

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3} + \sqrt{31,960}}{9.8}$ s. Thus the velocity at impact is

$$\vec{v}\left(\frac{100\sqrt{3} + \sqrt{31,960}}{9.8}\right) = \left\langle 100, 100\sqrt{3} - 9.8\left(\frac{100\sqrt{3} + \sqrt{31,960}}{9.8}\right) \right\rangle = \langle 100, -\sqrt{31,960} \rangle \text{ and the speed is}$$

$$\|\vec{v}\| = \sqrt{10,000 + 31,960} = \sqrt{41,960} \approx 205 \text{ m/s.}$$

Problem code: REHQS

3. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?

Place the ball at the origin and consider \mathbf{j} to be pointing in the northward direction with \mathbf{i} pointing east and \mathbf{k} pointing upward. Force = mass \times acceleration \Rightarrow acceleration = force/mass, so the wind applies a constant acceleration of $4\text{ N}/0.8\text{ kg} = 5\text{ m/s}^2$ in the easterly direction.

Combined with the acceleration due to gravity, the acceleration acting on the ball is the 3D vector $\mathbf{a}(t) = \langle 5, 0, -9.8 \rangle$. Then $\vec{v}(t)$ is the anti-derivative of $\vec{a}(t)$, or $\vec{v}(t) = \langle 5t, 0, -9.8t \rangle + \vec{c}$ where \vec{c} is a constant vector.

$$\begin{aligned}\text{We know } \vec{v}(0) &= \vec{c} = \langle 0, -30 \cos 30^\circ, 30 \sin 30^\circ \rangle \\ &= \langle 0, -15\sqrt{3}, 15 \rangle \\ \text{so } \vec{c} &= \langle 0, -15\sqrt{3}, 15 \rangle \text{ and} \\ \vec{v}(t) &= \langle 5t, -15\sqrt{3}, (15 - 9.8t) \rangle\end{aligned}$$

Finding the position vector over time, we anti-differentiate the velocity.

$$\begin{aligned}\mathbf{r}(t) &= \langle 2.5t^2, -15\sqrt{3}t, (15t - 4.9t^2) \rangle + \vec{d} \\ \text{but } \mathbf{r}(0) &= \vec{d} = \mathbf{0}, \text{ so} \\ \mathbf{r}(t) &= \langle 2.5t^2, -15\sqrt{3}t, (15t - 4.9t^2) \rangle.\end{aligned}$$

The ball lands when the z coordinate reaches 0, or $15t - 4.9t^2 = 0$. Solving for t , this occurs when $t = 0$ or $t = 15/4.9 \approx 3.0612$ s, so the ball lands at approximately

$\mathbf{r}(3.0612) \approx \langle 23.43, -79.53, 0 \rangle$ which is 82.9 m away in the direction S 16.4° E.

Its speed is approximately $|\vec{v}(3.0612)| \approx |\langle 15.306, -15\sqrt{3}, -15 \rangle| \approx 33.68$ m/s.

Problem code: EQWXN

4. Consider the target practice problem where you control the launch angle of a projectile, and you are trying select the angle so that the projectile lands at a specific location.
- Assuming that the only force acting on the projectile is gravity, find the formula for the 2D vector-valued velocity if the object is launched with an elevation angle θ from horizontal and an initial speed of 15 m/s.
 - Based on your previous answer, find a formula for the position of the object over time, given that it is launched from coordinates $(0,0)$.
 - Find the **two** possible launch angles θ_1 and θ_2 in degrees (between 0 and 90) that would lead to the projectile landing at the point $(22.5, 0)$.
The double-angle formula $\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$ may be helpful.

- (a) Given the acceleration is only due to gravity, the acceleration vector would be $\vec{a}(t) = \langle 0, -9.8 \rangle$ m/s². The velocity vector that must have given rise to this is $\vec{v}(t) = \langle k_1, -9.8t + k_2 \rangle$, for some constants k_1 and k_2 . (Check by differentiating \vec{v} and seeing it equals our \vec{a} .)

We can use the initial speed of 15 m/s and the launch angle θ to determine the values of k_1 and k_2 . At time $t = 0$, the velocity vector will be $\vec{v}(0) = \langle 15 \cos(\theta), 15 \sin(\theta) \rangle$, so:

$$\begin{aligned}k_1 &= 15 \cos(\theta) \text{ and} \\ k_2 &= 15 \sin(\theta)\end{aligned}$$

Thus the velocity function will be

$$\vec{v}(t) = \langle 15 \cos(\theta), -9.8t + 15 \sin(\theta) \rangle$$

- (b) To find the position, we undo the **time** derivative again from the velocity. I.e. any angle is a launch angle and is constant with respect to time.
The position $\vec{r}(t)$ must be

$$\vec{r}(t) = \langle 15 \cos(\theta) \cdot t + k_3, -9.8 \frac{t^2}{2} + 15 \sin(\theta) \cdot t + k_4 \rangle$$

for some constants k_3 and k_4 . (Again, we can check this: if we differentiate $\vec{r}(t)$ with respect to the variable t , we will get our earlier formula for $\vec{v}(t)$.)

Since we know $\vec{r}(0) = (0, 0)$ at the initial time $t = 0$, and subbing in $t = 0$ into our formula gives

$$\begin{aligned} \vec{r}(0) &= \langle 15 \cos(\theta) \cdot 0 + k_3, -9.8 \frac{0^2}{2} + 15 \sin(\theta) \cdot 0 + k_4 \rangle \\ &= \langle k_3, k_4 \rangle \end{aligned}$$

that means that we have $\langle 0, 0 \rangle = \langle k_3, k_4 \rangle$, so both constants must equal 0.

Our final position formula then is

$$\vec{r}(t) = \langle 15 \cos(\theta) \cdot t, -9.8 \frac{t^2}{2} + 15 \sin(\theta) \cdot t \rangle$$

- (c) To find the landing point, we set $y = 0$ in the position and solve for the resulting x coordinate for the position.

$$\begin{aligned} y = 0 &\Rightarrow -9.8 \frac{t^2}{2} + 15 \sin(\theta) \cdot t = 0 \\ &t(-4.9t + 15 \sin(\theta)) = 0 \\ &t = 0 \text{ or } 4.9t = 15 \sin(\theta) \\ &t = \frac{15 \sin(\theta)}{4.9} \end{aligned}$$

We ignore the $t = 0$ point because it is the launch point: we know the x there already is 0, not our target $x = 22.5$.
Substituting the non-zero t value into the expression for the x position:

$$\begin{aligned} x &= 15 \cos(\theta) \cdot t = 15 \cos(\theta) \left(\frac{15 \sin(\theta)}{4.9} \right) \\ &= \frac{15^2}{4.9} \cos(\theta) \sin(\theta) \end{aligned}$$

This is hard to solve with two θ 's in the formula, so we use re-arrange the double angle formula hint given in the question to $\frac{1}{2} \sin(2\theta) = \cos(\theta) \sin(\theta)$:

$$\begin{aligned} x &= \frac{15^2}{4.9} \left(\frac{1}{2} \sin(2\theta) \right) \\ &= \frac{15^2}{9.8} \sin(2\theta) \end{aligned}$$

Solving for the angle that leads to a landing x of exactly 22.5 meters,

$$22.5 = \frac{15^2}{9.8} \sin(2\theta)$$

$$0.98 = \sin(2\theta)$$

$$\arcsin(0.98) = 1.37 \approx 2\theta \text{ is one inverse sine value in radians}$$

$$\theta_1 \approx 0.685 \text{ rad} \approx 39.25 \text{ degrees.}$$

But that is only one possible angle. From the inverse sine step, and the symmetry in the sine function that $\sin(\alpha) = \sin(\pi - \alpha)$, we could find a second solution:

$$0.98 = \sin(2\theta) \text{ also means}$$

$$(\pi - 1.37) \approx 2\theta$$

$$\theta_2 \approx 0.89 \text{ rad} = 50.75 \text{ degrees.}$$

Two angles that will land the projectile at $(22.5, 0)$ are $\theta_1 = 39.25^\circ$ and $\theta_2 = 50.75^\circ$.

Inverse Trigonometry

5. Find the exact value of the following expressions:

- | | |
|--------------------------------|--|
| (i) $\arcsin(\sqrt{3}/2)$ | (vii) $\operatorname{arccot}(-\sqrt{3})$ |
| (ii) $\arccos(-1)$ | (viii) $\arccos(-\frac{1}{2})$ |
| (iii) $\arctan(1/\sqrt{3})$ | (ix) $\tan(\arctan 10)$ |
| (iv) $\operatorname{arcsec} 2$ | (x) $\arcsin(\sin(7\pi/3))$ |
| (v) $\arctan 1$ | (xi) $\tan(\operatorname{arcsec} 4)$ |
| (vi) $\arcsin(1/\sqrt{2})$ | (xii) $\sin(2 \arcsin(\frac{3}{5}))$ |

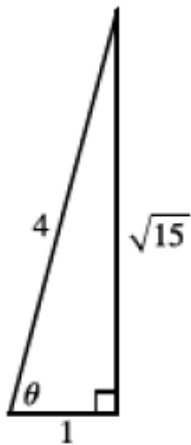
- (i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- (ii) $\cos^{-1}(-1) = \pi$ since $\cos \pi = -1$ and π is in $[0, \pi]$.
- (iii) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ since $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\frac{\pi}{6}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- (iv) $\sec^{-1} 2 = \frac{\pi}{3}$ since $\sec \frac{\pi}{3} = 2$ and $\frac{\pi}{3}$ is in $[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$.
- (v) $\arctan 1 = \frac{\pi}{4}$ since $\tan \frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- (vi) $\sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$ since $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\frac{\pi}{4}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- (vii) $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$ since $\cot \frac{5\pi}{6} = -\sqrt{3}$ and $\frac{5\pi}{6}$ is in $(0, \pi)$.
- (viii) $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$ since $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\frac{2\pi}{3}$ is in $[0, \pi]$.
- (ix) In general, $\tan(\arctan x) = x$ for any real number x . Thus, $\tan(\arctan 10) = 10$.
- (x) $\arcsin(\sin \frac{7\pi}{3}) = \arcsin(\sin \frac{\pi}{3}) = \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- (Recall that $\frac{7\pi}{3} = \frac{\pi}{3} + 2\pi$ and the sine function is periodic with period 2π .)
- (xi) Let θ denote the angle whose secant is 4, that is, $\theta = \sec^{-1} 4$, as shown in the figure.

Thus, $\tan(\sec^{-1} 4) = \tan \theta = \frac{\sqrt{15}}{1} = \sqrt{15}$.

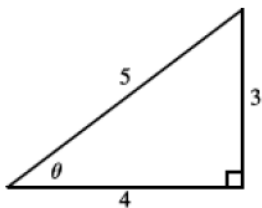
Alternative solution:

$$\sec^2 \theta = 1 + \tan^2 \theta \Rightarrow 4^2 = 1 + \tan^2 \theta \Rightarrow \tan \theta = \sqrt{15}$$

(Note that $\sec \theta = 4 > 0 \Rightarrow \theta$ is in $[0, \frac{\pi}{2}) \Rightarrow \tan \theta > 0$.)



- (xii) Let $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ [see figure].
 $\sin\left(2 \sin^{-1}\left(\frac{3}{5}\right)\right) = \sin(2\theta) = 2 \sin \theta \cos \theta = 2 \left(\frac{3}{5}\right) \left(\frac{4}{5}\right) = \frac{24}{25}$



Problem code: LCUHA

6. Prove that $\cos(\arcsin x) = \sqrt{1 - x^2}$.

Let $y = \arcsin x$. Then $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \Rightarrow \cos y \geq 0$, so $\cos(\arcsin x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

Problem code: FSFNT (Video Solution by R.K.)

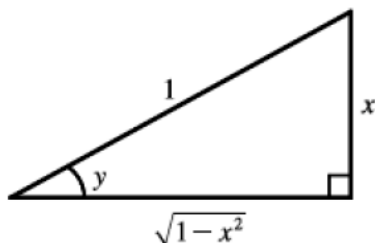
7. Simplify the following expressions:

(a) $\tan(\arcsin x)$

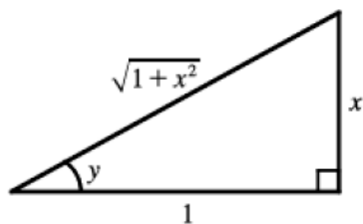
(b) $\sin(\arctan x)$

(c) $\cos(2 \arctan x)$. The double-angle formula $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ may be helpful.

(a) Let $y = \arcsin x$. Then $\sin y = x$, so from the triangle we see that $\tan(\arcsin x) = \tan y = \frac{x}{\sqrt{1 - x^2}}$



(b) Let $y = \arctan x$. Then $\tan y = x$, so from the triangle we see that $\sin(\arctan x) = \sin y = \frac{x}{\sqrt{1 + x^2}}$



(c) Let $y = \arctan x$. Then $\tan y = x$, so we have the same triangle as part (b) and we get $\cos(2 \arctan x) = \cos 2y = \cos^2 y - \sin^2 y = \left(\frac{1}{\sqrt{1 + x^2}}\right)^2 - \left(\frac{x}{\sqrt{1 + x^2}}\right)^2 = \frac{1 - x^2}{1 + x^2}$.

Problem code: JGVHG (Video Solution by T.N.)

8. Find the domain and range of the function $g(x) = \arcsin(3x + 1)$.

Domain (g) = $x \mid -1 \leq 3x + 1 \leq 1 = x \mid -1 \leq 3x \leq 0 = x \mid -\frac{2}{3} \leq x \leq 0 = [-\frac{2}{3}, 0]$.

Range (g) = $y \mid -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Problem code: BDLEX (Video Solution by H.G.)

9. Let $f(x) = \arcsin(x)$.

- (a) Compute $f'(x)$
- (b) Find $f'(0.4)$.

Recall that the derivative of $\arcsin(x)$ is:

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

Evaluating this at $x = 0.4$, we get:

$$f'(0.4) \approx 1.0911$$

Problem code: BVTAG ([Video Solution by D. C.](#))

10. Let $f(x) = \frac{\arccos(14x)}{\arcsin(14x)}$. Compute $f'(x)$.

Applying the quotient rule:

$$\begin{aligned} f'(x) &= \frac{\arcsin(14x) (\arccos(14x))' - \arccos(14x) (\arcsin(14x))'}{(\arcsin(14x))^2} \\ &= \frac{\frac{-14 \arcsin(14x)}{\sqrt{1-(14x)^2}} - \frac{14 \arccos(14x)}{\sqrt{1-(14x)^2}}}{(\arcsin(14x))^2} \\ &= \frac{-14 (\arcsin(14x) + \arccos(14x))}{(\arcsin(14x))^2 \sqrt{1-(14x)^2}} \end{aligned}$$

Problem code: HUWFFV

Interpreting Derivatives

11. The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of the points.

$y = x^3 - 9x^2 - 16x + 1$ has slopes given by $y' = 3x^2 - 18x - 16$. The original curve will have a slope of 5 at points where $y' = 5$:

$$\begin{aligned} \text{Solve for } x: \quad 5 &= 3x^2 - 18x - 16 \\ 0 &= 3x^2 - 18x - 21 \\ 0 &= x^2 - 6x - 7 \\ 0 &= (x - 7)(x + 1) \\ x &= 7, -1 \end{aligned}$$

The function will have slopes of 5 at $x = 7$ and $x = -1$. The coordinates of these points are (using $f(x)$), $(-1, 7)$ and $(7, -209)$.

Problem code: EQNUW ([Video Solution by K.J.](#))

12. Determine coefficients a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 3$ and $p'(1) = 1$.

Let $p(x) = x^2 + ax + b$ satisfy $p(1) = 3$ and $p'(1) = 1$. Since $p'(x) = 2x + a$, this implies $3 = p(1) = 1 + a + b$ and $1 = p'(1) = 2 + a$; i.e., $a = -1$ and $b = 3$.

Problem code: QQAJH ([Video Solution by C.C.](#))

13. A ball is thrown up in the air, and its height over time is given by

$$f(t) = -4.9t^2 + 25t + 3$$

where t is in seconds and $f(t)$ is in meters.

- (a) What is the average velocity of the ball during the first two seconds? Include units in your answer.
- (b) Find the instantaneous velocity of the ball at $t = 2$.
- (c) Compute the acceleration of the ball at $t = 2$.
- (d) What is the highest height reached by the ball?
- (e) How long is the ball in the air?

- (a) The **average** velocity is not found directly with derivatives, but rather by taking total distance

traveled, divided by time taken.

$$f(0) = 3 \text{ meters}$$

$$f(2) = 33.4 \text{ meters}$$

so avg speed is

$$= \frac{\Delta f}{\Delta t} = \frac{f(2) - f(0)}{2 - 0} = \frac{(33.4 - 3) \text{ meters}}{2 \text{ seconds}} = 15.2 \text{ m/s}$$

- (b) The **instantaneous** velocity at $t = 2$ is given by the derivative.

$$f'(t) = -9.8t + 25$$

$$\text{at } t = 2, \quad f'(2) = 5.4 \text{ m/s}$$

- (c) The acceleration at $t = 2$ is given by the **second** derivative of position,

$$f''(t) = -9.8$$

$$\text{so at } t = 2 \text{ (and all } t \text{ values), } f''(2) = -9.8 \text{ m/s}^2$$

- (d) You can use parabola analysis to find vertex of the parabola. A more calculus-oriented method is to find when the velocity of the ball is zero: that will be at the top of its arc.

$$f'(t) = -9.8t + 25$$

$$\text{Setting } f' \text{ to zero, } \quad 0 = -9.8t + 25$$

$$t \approx 2.551$$

at the top of its arc. The height at this point is $f(2.551) \approx 34.89$ m.

- (e) Knowing that the height is defined by a parabola, and given that the ball rose for $t \approx 2.551$ seconds means it will also take ≈ 2.551 seconds to return to earth, so it spends $2 \cdot (2.551) \approx 5.102$ seconds in the air. **Note:** this is not exactly correct, though, because the ball was launched ($t = 0$) from height $f(0) = 3$, but the impact occurs when $f(t) = 0$ later. It will take 5.102 second to return to height 3 m, but just a little bit longer to actually impact the ground at height 0 m.

A more accurate approach to find the impact time is to use the quadratic formula to find the two roots of $f(t)$, the larger of which will be the landing time:

$$f(t) = -4.9t^2 + 25t + 3 \text{ equals zero when}$$

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(3)}}{2(-4.9)}$$

$$= 5.2193 \text{ and } -0.1173$$

The impact will be at exactly $t = 5.2193$ seconds, or just little longer than the estimate of 5.102 we found using the parabola insight.

Problem code: HFGXM

14. The height of a sand dune (in centimeters) is represented by $f(t) = 800 - 5t^2$ cm, where t is measured in years since 1995. Find the values $f(8)$ and $f'(8)$, including units, and determine what each means in terms of the sand dune.

Since $f(t) = 800 - 5t^2$ cm, $f(8) = 800 - 5(8)^2 = 480$ cm. Since $f'(t) = -10t$ cm/yr, we have $f'(8) = -10(8) = -80$ cm/yr. In the year 2003, the sand dune was 480 cm high and it was eroding at a rate of 80 centimeters per year.

Problem code: VBZPR

15. With a yearly inflation rate of 5%, prices are given by

$$P(t) = P_0(1.05)^t$$

where P_0 is the price in dollars when $t = 0$ and t is time in years. Suppose $P_0 = 1$. How fast (in cents per year) are prices rising when $t = 10$?

$P = P_0(1.05)^t$. If $P_0 = 1$, then $P'(t) = (1.05)^t \ln(1.05)$. At $t = 0$, $P'(0) = (1.05)^0 \ln(1.05) = \ln(1.05) \approx 0.0488$ dollars/year.

The prices are increasing at a rate of approximately 0.0488 dollars per year initially (at $t = 0$), or 4.88 cents per year.

At $t = 10$, $P'(10) = (1.05)^{10} \ln(1.05) \approx 0.0795$ dollars/year, or 7.95 cents per year.

Problem code: BDWWW

16. With t in years since January 1st, 1990, the population P of a small US town has been given by

$$P = 35,000(0.98)^t$$

At what rate was the population changing on January 1st, 2010, in units of people/year?

$P(t) = 35000(0.98)^t$, so $P'(t) = 35000(0.98)^t \ln(0.98)$

Since $t = 0$ is January 1st 1990, and t is in years, Jan 1st 2010 represents $t = 20$.

$P'(20) = 35000(0.98)^{20} \ln(0.98) \approx -472$ people/year. The rate is negative, indicating that people are leaving the town.

Problem code: ANVWT

17. The value of an automobile can be approximated by the function

$$V(t) = 25(0.85)^t,$$

where t is in years from the date of purchase, and $V(t)$ is its value, in thousands of dollars.

- Evaluate and interpret $V(4)$.
- Find an expression for $V'(t)$.
- Evaluate and interpret $V'(4)$.

- (a) $V(4) \approx 13.05$. 4 years after purchase, the car will be worth approximately 13 thousand dollars.
- (b) $V'(t) = 25(0.85)^t(\ln(0.85)) \approx -4.06(0.85)^t$.
- (c) $V'(4) = -2.12$ means that, 4 years after purchase, the car will be losing value at a rate of roughly 2 thousand dollars per year.

Problem code: CTNHS

18. The theory of relativity predicts that an object whose mass is m_0 when it is at rest will appear heavier when moving at speeds near the speed of light. When the object is moving at speed v , its mass m is given by

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}, \text{ where } c \text{ is the speed of light}$$

(a) Find $\frac{dm}{dv}$.

(b) In terms of physics, what does $\frac{dm}{dv}$ tell you?

$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$, and m_0 and c are constants. We can rewrite this using powers as $m = m_0 \left[1 - \frac{1}{c^2}v^2\right]^{-1/2}$

(a)

$$\begin{aligned} \frac{dm}{dv} &= m_0 \frac{-1}{2} \left[1 - \frac{1}{c^2}v^2\right]^{-3/2} \left(\frac{-1}{c^2}(2v)\right) \\ &= \frac{m_0}{c^2}v \left[1 - \frac{1}{c^2}v^2\right]^{-3/2} \end{aligned}$$

(b) The derivative $\frac{dm}{dv}$ represents how the (relativistic) mass of an object changes as its velocity changes.

Note that for small v ($v \ll c$ or “ v much smaller than c ”), this rate is almost zero, since relativistic effects are only pronounced when our speed approaches the speed of light.

Also note the sign of the derivative: for any v less than the speed of light c , we will have $\frac{v^2}{c^2} < 1$, or $1 - \frac{v^2}{c^2}$ positive. This means that all the factors in the derivative are **positive**:

$$\frac{dm}{dv} \underbrace{\frac{m_0}{c^2}v}_{\text{pos.}} \left[\underbrace{1 - \frac{1}{c^2}v^2}_{\text{pos.}} \right]^{-3/2}$$

A positive $\frac{dm}{dv}$ derivative indicates that as speed v increases, so too does mass m .

Problem code: RHKKA

19. (a) Find the *eighth* derivative of $f(x) = x^7 + 5x^5 - 4x^3 + 6x - 7$. Look for patterns as you go...

(b) Find the *seventh* derivative of $f(x)$.

(a) Since each derivative removes one power from a polynomial, the eighth derivative of any seventh-degree polynomial will be zero.

(b) Only the derivative of the x^7 term will survive seven derivatives. Looking at the pattern,

$$f'(x) = 7x^6 + \dots$$

$$f''(x) = 7 \cdot 6x^5 + \dots$$

\vdots

$$f^{(7)} = 7 \cdot 6 \cdot 5 \cdot \dots \cdot 1x^0 = 7! \text{ (7 factorial)} = 5040$$

Problem code: FGCCR

Linear Approximations and Tangent Lines

20. Find the equation of the tangent line to the graph of f at $(1,1)$, where f is given by $f(x) = 2x^3 - 2x^2 + 1$.

$$f(x) = 2x^3 - 2x^2 + 1.$$

In general, the slopes of the function are given by $f'(x) = 6x^2 - 4x$

At the point $(1, 1)$ (which you should check is actually on the graph of $f(x)$!), the slope is

$$f'(1) = 6 - 4 = 2$$

Using the point/slope formula for a line (or the tangent line formula), a line tangent to the graph of $f(x)$ at the

point $(1, 1)$ is

$$y = f'(1)(x - 1) + f(1)$$

$$= 2(x - 1) + 1$$

$$\text{or } y = 2x - 1$$

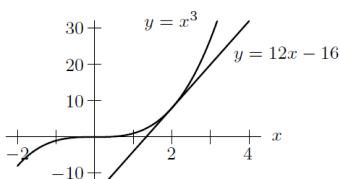
Problem code: SLJDR (Video Solution by H.G.)

21. (a) Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$.
- (b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate $f(x) = x^3$ would produce *over-* or *under-*estimates of $f(x)$ near $x = 2$.

- (a) $f(x) = x^3$, so $f'(x) = 3x^2$.
At $x = 2$, $f(2) = 8$ and $f'(2) = 12$, so the tangent line to $f(x)$ at $x = 2$ is

$$y = 12(x - 2) + 8$$

(b)



From the graph of $y = x^3$, it is clear that the tangent line at $x = 2$ will lie *below* the actual curve. This means that using the tangent line to estimate $f(x)$ values will produce *underestimates* of $f(x)$.

Problem code: WPWHK

22. Given a power function of the form $f(x) = ax^n$, with $f'(3) = 16$ and $f'(6) = 128$, find n and a .

Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(3) = (an)3^{n-1} = 16$, and $f'(6) = (an)6^{n-1} = 128$. Therefore,

$$\frac{f'(6)}{f'(3)} = \frac{128}{16} = 8.$$

But

$$\frac{f'(6)}{f'(3)} = \frac{(an)6^{n-1}}{(an)3^{n-1}} = 2^{n-1},$$

so $2^{n-1} = 8$, and so $n = 4$.

Substituting $n = 4$ into the expression for $f'(3)$, we get $4a3^3 = 16$, so $a = \frac{4}{27}$.

Problem code: CWUUY ([Video Solution by K.J.](#))

23. Find all values of x where the tangent lines to $y = x^8$ and $y = x^9$ are parallel.

Let $f(x) = x^8$ and let $g(x) = x^9$. The two graphs have parallel tangent lines at all x where $f'(x) = g'(x)$.

$$f'(x) = g'(x)$$

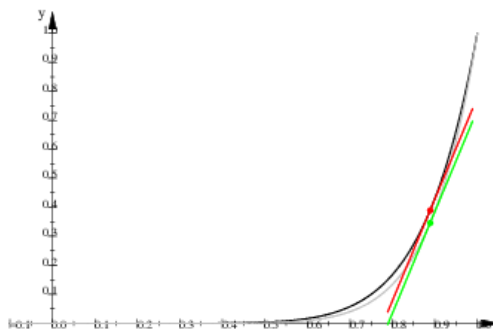
$$8x^7 = 9x^8$$

$$8x^7 - 9x^8 = 0$$

$$x^7(8 - 9x) = 0$$

hence, $x = 0$ or $x = \frac{8}{9}$.

The point at $x = 0$ is easy to visualize (both graphs are flat there). Here is a graph showing the parallel tangents at $x = 8/9$.



Problem code: DBQZW

24. Consider the function $f(x) = 9 - e^x$.

- (a) Find the slope of the graph of $f(x)$ at the point where the graph crosses the x -axis.
- (b) Find the equation of the tangent line to the curve at this point.
- (c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

- (a) $f(x) = 9 - e^x$ crosses the x -axis where $0 = 9 - e^x$, which happens when $e^x = 9$, so $x = \ln 9$. Since $f'(x) = -e^x$, $f'(\ln 9) = -9$.
- (b) $y = -9(x - \ln(9))$.
- (c) The slope of the normal line is the negative reciprocal of the slope of the tangent, so $y = \frac{1}{9}(x - \ln(9))$.

Problem code: LRURC

25. Consider the function $y = 2^x$.

- (a) Find the tangent line based at $x = 1$, and find where the tangent line will intersect the x axis.
- (b) Find the point on the graph $x = a$ where the tangent line will pass through the origin.

- (a) We find the linearization using $f(x) = 2^x$, so $f'(x) = 2^x \ln(2)$ (non- e exponential derivative rule).

At the point $x = 1$, $f(1) = 2^1 = 2$ and $f'(1) = 2^1 \ln(2)$, so the linear approximation to $f(x)$ is $L(x) = 2 + (2 \ln(2))(x - 1)$.

Solving for where this line intersects the x axis (or the $y = 0$ line), we find the x intercept is approximately -0.4427 .

- (b) This question is more general. Instead of asking for a linearization at a specific point, it is asking “at what point would the linearization pass through the origin?” Let us give the point a name: $x = a$ (as opposed to $x = 1$ used in part (a)).

From the function and the derivatives, the linearization at the point $x = a$ is given by:

$$L_a(x) = \underbrace{2^a}_{f(a)} + \underbrace{2^a \ln(2)}_{f'(a)}(x - a)$$

That is true in general, but we want the point $x = a$ where the linearization will go through $(0, 0)$, i.e. for which $L_a(0) = 0$:

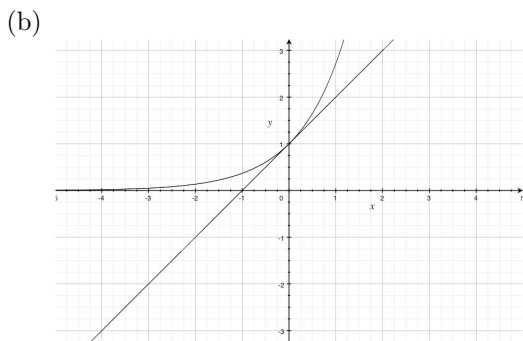
$$\begin{aligned} 0 &= 2^a + 2^a \ln(2)(0 - a) \\ \text{Solving for } a, \quad 0 &= 2^a(1 - a \ln(2)) \\ 0 &= 1 - a \ln(2) \\ a \ln(2) &= 1 \\ a &= \frac{1}{\ln(2)} \approx 1.442 \end{aligned}$$

At that x point, the graph of $y = 2^x$'s tangent line will pass exactly through the origin.

Problem code: CBPMY

26. (a) Find the tangent line approximation to $f(x) = e^x$ at $x = 0$.
 (b) Use a sketch of $f(x)$ and the tangent line to determine whether the tangent line produces over- or under-estimates of $f(x)$.
 (c) Use your answer from part (b) to decide whether the statement $e^x \geq 1 + x$ is always true or not.

- (a) $f(x) = e^x$, so $f'(x) = e^x$ as well. To build the tangent line at $x = 0$, we use $a = 0$ as our reference point: $f(0) = e^0 = 1$, and $f'(0) = e^0 = 1$. The tangent line is therefore $l(x) = 1(x - 0) + 1 = x + 1$



Since the exponential graph is concave up, it curves upwards away from the graph. This means that the linear approximation will always be an underestimate of the original function.

- (c) Since the linear function will always underestimate the value of e^x , we can conclude that

$$1 + x \leq e^x$$

, and they will be equal only at the tangent point, $x = 0$.

Problem code: HNDEF

27. The speed of sound in dry air is

$$f(T) = 331.3 \sqrt{1 + \frac{T}{273.15}} \text{ m/s}$$

where T is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0° C .

$$\begin{aligned} f(T) &= 331.3 \sqrt{1 + \frac{T}{273.15}} \\ f'(T) &= 331.3 \left(\frac{1}{2} \right) \left(\frac{1}{\sqrt{1 + \frac{T}{273.15}}} \right) \frac{1}{273.15} \end{aligned}$$

so at $T = 0^\circ \text{ C}$, $f(0) = 331.3$ $f'(0) = \frac{331.3}{(2)(273.15)} \approx 0.606$

Thus the speed of sound for air temperatures around 0° C is

$$f(T) \approx 0.606(T - 0) + 331.3, \text{ or } f(t) \approx 0.606T + 331.3 \text{ m/s}$$

Problem code: YVGHG

28. Consider the graphs of $y = \sin(x)$ (regular sine graph), and $y = ke^{-x}$ (exponential decay, but scaled vertically by k).
 If $k \geq 1$, the two graphs will intersect. What is the smallest value of k for which two graphs will be *tangent* at that intersection point?

Let $f(x) = \sin(x)$ and $g(x) = ke^{-x}$. They intersect when $f(x) = g(x)$, and they are tangent at that intersection if $f'(x) = g'(x)$ as well. Thus we must have

$$\sin(x) = ke^{-x} \quad \text{and} \quad \cos(x) = -ke^{-x}$$

We can't solve either equation on its own, but we can divide one by the other:

$$\begin{aligned} \frac{\sin(x)}{\cos(x)} &= \frac{ke^{-x}}{-ke^{-x}} \\ \tan(x) &= -1 \\ x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \dots \end{aligned}$$

Since we only need one value of k , we try the first value, $x = 3\pi/4$.

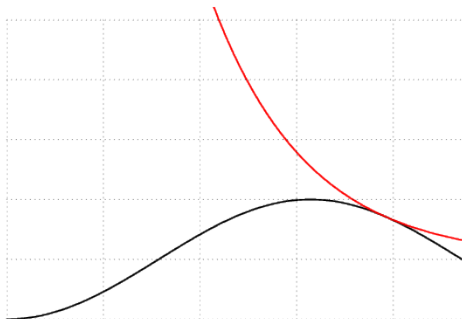
$$\begin{aligned}\sin(3\pi/4) &= ke^{-3\pi/4} \\ \frac{1}{\sqrt{2}}e^{3\pi/4} &= k \\ k &\approx 7.46\end{aligned}$$

We confirm our answer by verifying both the values and derivatives are equal at $x = 3\pi/4$,

$$\begin{aligned}\sin(3\pi/4) &= 7.46e^{-3\pi/4} \approx 0.7071 \text{ (same } y \text{: intersection)} \\ \text{and } \cos(3\pi/4) &= -7.46e^{-3\pi/4} \approx -0.7071 \text{ (same derivative)}\end{aligned}$$

The actual point of tangency is at $(x, y) = \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

A sketch is shown below.



Problem code: PLHVB (Video Solution by K. A.)

29. (a) Show that $1 + kx$ is the local linearization of $(1 + x)^k$ near $x = 0$.

(b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?

(a)

$$\begin{aligned}f(x) &= (1 + x)^k & f'(x) &= k(1 + x)^{k-1} \\ \text{so at } x = 0, & f(0) = 1^k = 1 & f'(0) &= k(1^{k-1}) = k\end{aligned}$$

so the tangent line at $x = 0$ will be

$$y = k(x - 0) + 1 \text{ or } y = 1 + kx$$

(b) As an estimate for the square root of 1.1, we could note that $\sqrt{1.1} = (1 + 0.1)^{1/2}$. This matches exactly the form of $f(0.1)$ if we choose $k = \frac{1}{2}$. From our linearization above,

$$f(0.1) \approx 1 + \frac{1}{2}(0.1) = 1.05$$

so yes, a good approximation for $\sqrt{1.1}$ is 1.05. (Calculator gives the value of ≈ 1.0488 .)

Problem code: HTLJR