Week #4 - Inverse Trig, Tangent Lines and Linearization

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Projectile Motion

1. A projectile is fired from ground level on a flat region with an initial speed of 200 m/s and angle of elevation 60°. Find:
   (a) The range of the projectile.
   (b) The maximum height reached.
   (c) The speed at impact.
2. Rework question 1 if the projectile is fired from a position 100 m above the ground.
3. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?
4. Consider the target practice problem where you control the launch angle of a projectile, and you are trying select the angle so that the projectile lands at a specific location.
   (a) Assuming that the only force acting on the projectile is gravity, find the formula for the 2D vector-valued velocity if the object is launched with an elevation angle θ from horizontal and an initial speed of 15 m/s.
   (b) Based on your previous answer, find a formula for the position of the object over time, given that it is launched from coordinates (0,0).
   (c) Find the **two** possible launch angles θ_1 and θ_2 in degrees (between 0 and 90) that would lead to the projectile landing at the point $(22.5, 0)$.
   The double-angle formula $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ may be helpful.

Inverse Trigonometry

5. Find the exact value of the following expressions:
   (i) $\arcsin(\sqrt{3}/2)$
   (ii) $\arccos(-1)$
   (iii) $\arctan(1/\sqrt{3})$
   (iv) $\text{arcsec } 2$
   (v) $\arctan 1$
   (vi) $\arcsin(1/\sqrt{2})$
   (vii) $\text{arccot } (-\sqrt{3})$
   ($viii$) $\arccos(-\frac{1}{2})$
   (ix) $\tan(\arctan 10)$
   (x) $\arcsin(\sin(7\pi/3))$
   (xi) $\tan(\text{arcsec } 4)$
   (xii) $\sin\left(2\arcsin(\frac{3}{5}\right)$
6. Prove that $\cos(\arcsin x) = \sqrt{1-x^2}$.
7. Simplify the following expressions:
   (a) $\tan(\arcsin x)$
   (b) $\sin(\arctan x)$
   (c) $\cos(2\arctan x)$. The double-angle formula $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ may be helpful.
8. Find the domain and range of the function $g(x) = \arcsin(3x + 1)$.
9. Let $f(x) = \arcsin(x)$.
   (a) Compute $f'(x)$
   (b) Find $f'(0.4)$.
10. Let $f(x) = \dfrac{\arccos(14x)}{\arcsin(14x)}$. Compute $f'(x)$.

Interpreting Derivatives

11. The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of the points.

12. Determine coefficients a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 3$ and $p'(1) = 1$.

13. A ball is thrown up in the air, and its height over time is given by
$$f(t) = -4.9t^2 + 25t + 3$$
where t is in seconds and $f(t)$ is in meters.

    (a) What is the average velocity of the ball during the first two seconds? Include units in your answer.

    (b) Find the instantaneous velocity of the ball at $t = 2$.

    (c) Compute the acceleration of the ball at $t = 2$.

    (d) What is the highest height reached by the ball?

    (e) How long is the ball in the air?

14. The height of a sand dune (in centimeters) is represented by $f(t) = 800 - 5t^2$ cm, where t is measured in years since 1995. Find the values $f(8)$ and $f'(8)$, including units, and determine what each means in terms of the sand dune.

15. With a yearly inflation rate of 5%, prices are given by
$$P(t) = P_0(1.05)^t$$
where P_0 is the price in dollars when $t = 0$ and t is time in years. Suppose $P_0 = 1$. How fast (in cents per year) are prices rising when $t = 10$?

16. With t in years since January 1st, 1990, the population P of a small US town has been given by
$$P = 35,000(0.98)^t$$
At what rate was the population changing on January 1st, 2010, in units of people/year?

17. The value of an automobile can be approximated by the function
$$V(t) = 25(0.85)^t,$$
where t is in years from the date of purchase, and $V(t)$ is its value, in thousands of dollars.

    (a) Evaluate and interpret $V(4)$.

    (b) Find an expression for $V'(t)$.

    (c) Evaluate and interpret $V'(4)$.

18. The theory of relativity predicts that an object whose mass is m_0 when it is at rest will appear heavier when moving at speeds near the speed of light. When the object is moving at speed v, its mass m is given by
$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}},$$ where c is the speed of light

    (a) Find $\frac{dm}{dv}$.

    (b) In terms of physics, what does $\frac{dm}{dv}$ tell you?

19. (a) Find the *eighth* derivative of $f(x) = x^7 + 5x^5 - 4x^3 + 6x - 7$. Look for patterns as you go...

    (b) Find the *seventh* derivative of $f(x)$.

Linear Approximations and Tangent Lines

20. Find the equation of the tangent line to the graph of f at (1,1), where f is given by $f(x) = 2x^3 - 2x^2 + 1$.

21. (a) Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$.

    (b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate $f(x) = x^3$ would produce *over*- or *under*-estimates of $f(x)$ near $x = 2$.

22. Given a power function of the form $f(x) = ax^n$, with $f'(3) = 16$ and $f'(6) = 128$, find n and a.

23. Find all values of x where the tangent lines to $y = x^8$ and $y = x^9$ are parallel.

24. Consider the function $f(x) = 9 - e^x$.

    (a) Find the slope of the graph of $f(x)$ at the point where the graph crosses the x-axis.

    (b) Find the equation of the tangent line to the curve at this point.

    (c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

25. Consider the function $y = 2^x$.

    (a) Find the tangent line based at $x = 1$, and find where the tangent line will intersect the x axis.

    (b) Find the point on the graph $x = a$ where the tangent line will pass through the origin.

26. (a) Find the tangent line approximation to $f(x) = e^x$ at $x = 0$.

    (b) Use a sketch of $f(x)$ and the tangent line to determine whether the tangent line produces over- or under-estimates of $f(x)$.

(c) Use your answer from part (b) to decide whether the statement $e^x \geq 1 + x$ is always true or not.

27. The speed of sound in dry air is

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}}\,\text{m/s}$$

where T is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0^o C.

28. Consider the graphs of $y = \sin(x)$ (regular sine graph), and $y = ke^{-x}$ (exponential decay, but scaled vertically by k).

If $k \geq 1$, the two graphs will intersect. What is the smallest value of k for which two graphs will be *tangent* at that intersection point?

29. (a) Show that $1+kx$ is the local linearization of $(1+x)^k$ near $x = 0$.

(b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?