Week #3 - Differentiation of Vector-Valued Functions

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Derivatives of Vector Functions

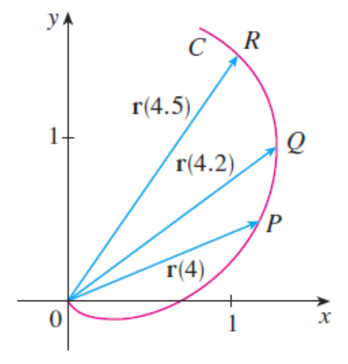
1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.

   (a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.

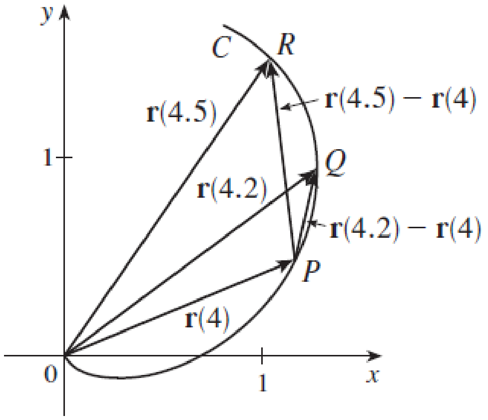
   (b) Draw the vectors $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5}$ and $\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$

   (c) Write expressions for $\mathbf{r}'(t)$ and the unit tangent vector $\mathbf{T}(4)$.

   (d) Draw the vector $\mathbf{T}(4)$.



(a)

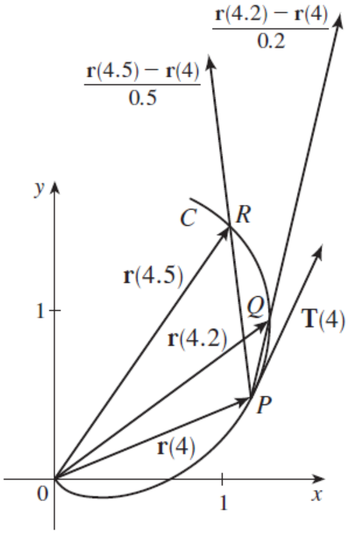


(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2\,[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$.
$\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5\,[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.

(c) By definition 1 (Stewart 7th, pg. 871), $\mathbf{r}'(4) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(4+h) - \mathbf{r}(4)}{h}$.

The **unit** (= length 1) tangent vector is $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{||\mathbf{r}'(4)||}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$, but with length 1.

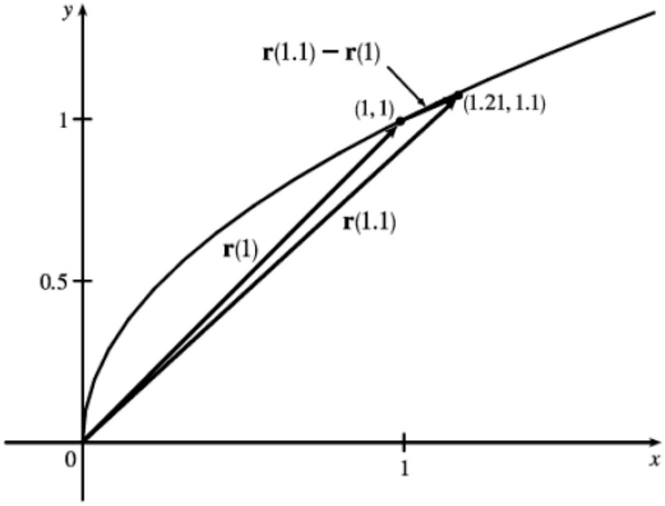


Problem code: RGSMT

2. (*a*) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t \rangle, \quad 0 \leq t \leq 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.

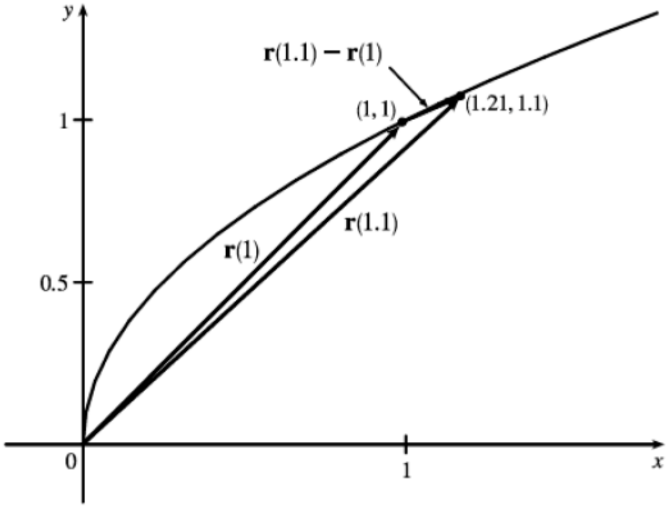
   (*b*) Draw the vector $\mathbf{r}'(1)$ starting at $(1, 1)$, and compare it with the vector $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$. Explain why these vectors are so close to each other in length and direction.

(a) The curve can be represented by the parametric equations $x = t^2, y = t, 0 \leq t \leq 2$. Eliminating the parameter, we have $x = y^2, 0 \leq 2 \leq 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1), \mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.



(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10 \cdot \langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$



As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be $\lim_{h\to 0} \dfrac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$. Since h is close to 0, we would expect $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

Problem code: FGCGU

3. For the vector equation $\mathbf{r}(t) = \langle t - 2, t^2 + 1 \rangle$

   (a) Sketch the plane curve.

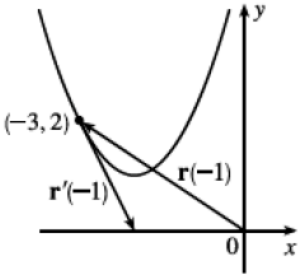
   (b) Find $\mathbf{r}'(t)$.

   (c) Sketch the position vector $\mathbf{r}(t)$ and the velocity vector $\mathbf{r}'(t)$ for $t = -1$.

(a) Sketch can be done using either a table of values, or by eliminating the parameter.
Eliminating the parameter: since $(x + 2)^2 = t^2 = y - 1 \;\Rightarrow y = (x + 2)^2 + 1$, the curve is a parabola.

(b) $\mathbf{r}'(t) = \langle 1, 2t \rangle$, $\mathbf{r}'(-1) = \langle 1, -2 \rangle$

(c)



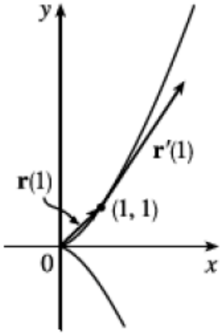
Problem code: XWFKA (Video Solution by K.M.)

4. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $t = 1$.

(a) Sketch can be done using either a table of values, or by eliminating the parameter.
Eliminating the parameter: since $x = t^2 = (t^3)^{2/3} = y^{2/3}$, the curve is the graph of $x = y^{2/3}$. Refer to (c) for the sketch.

(b) $\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle$, $\mathbf{r}'(1) = \langle 2, 3 \rangle$

(c)



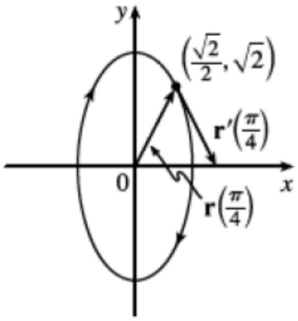
Problem code: NFMTF (Video Solution by K.M.)

5. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle \sin t, 2\cos t \rangle, \quad t = \pi/4$.

(a) Since $x = \sin(t), y = 2\cos(t)$, so $x^2 + (y/2)^2 = 1$. This relation (close to the formula $x^2 + y^2 = r^2$ for a circle) defines an *ellipse*. Refer to (c) for the sketch.

(b) $\mathbf{r}'(t) = \langle \cos t, -2\sin t \rangle, \quad \text{so } \mathbf{r}'(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2}, -\sqrt{2} \rangle$

(c)



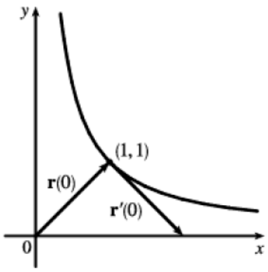
Problem code: ZLFGY (Video Solution by K.M.)

6. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle e^t, e^{-t} \rangle, \quad t = 0$.

(a) Since $y = e^{-t} = \dfrac{1}{e^t} = \dfrac{1}{x}$, the curve is part of the hyperbola $y = \dfrac{1}{x}$. Note that $x > 0, y > 0$ because both $x = e^t$ and $y = e^{-t}$ give positive values for all t. Refer to (c) for the sketch.

(b) $\mathbf{r}'(t) = \langle e^t - e^{-t} \rangle, \ \mathbf{r}'(0) = \langle 1, -1 \rangle$

(c)



Problem code: LQJVF

7. Find the derivative of the following vector functions.

   (a) $\mathbf{r}(t) = \langle t\sin(t), t^2, t\cos(2t)\rangle$.

   (b) $\mathbf{r}(t) = \langle \tan(t), \sec(t), 1/t^2\rangle$

   (c) $\mathbf{r}(t) = \langle t, 1, 2\sqrt{t}\rangle$

   (d) $\mathbf{r}(t) = \langle \frac{1}{1+t}, \frac{t}{1+t}, \frac{t^2}{1+t}\rangle$

   (e) $\mathbf{r}(t) = \langle e^{t^2}, -1, \ln(1+3t)\rangle$

   (f) $\mathbf{r}(t) = \langle a\ t\cos(3t), b\sin^3(t), c\cos^3 t\rangle$, where a, b and c are constants.

(a) $\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[t\sin t\right], \frac{d}{dt}\left[t^2\right], \frac{d}{dt}\left[t\cos 2t\right]\rangle = \langle t\cos t + \sin t, 2t, 2t(-\sin 2t) + \cos 2t \right\rangle$

$= \langle t\cos t + \sin t, 2t, \cos 2t - 2t\sin 2t\rangle$

(b) $\mathbf{r}(t) = \langle \tan t, \sec t, 1/t^2\rangle \ \Rightarrow\ \mathbf{r}'(t) = \langle \sec^2 t, \sec t\tan t, -2/t^3\rangle$

(c) $\mathbf{r}(t) = \langle t, 1, 2\sqrt{t}\rangle \ \Rightarrow\ \mathbf{r}'(t) = \left\langle 1, 0, 2\left(\tfrac{1}{2}t^{-1/2}\right)\right\rangle = \left\langle 1, 0, \frac{1}{\sqrt{t}}\right\rangle$

(d) $\mathbf{r}(t) = \left\langle \frac{1}{1+t}, \frac{t}{1+t}, \frac{t^2}{1+t}\right\rangle \ \Rightarrow$

$$\mathbf{r}'(t) = \left\langle \frac{0-1(1)}{(1+t)^2}, \frac{(1+t)\cdot 1 - t(1)}{(1+t)^2}, \frac{(1+t)\cdot 2t - t^2(1)}{(1+t)^2}\right\rangle = \left\langle -\frac{1}{(1+t)^2}, \frac{1}{(1+t)^2}, \frac{t^2+2t}{(1+t)^2}\right\rangle$$

(e) $\mathbf{r}(t) = \left\langle e^{t^2}, -1, \ln(1+3t)\right\rangle \ \Rightarrow\ \mathbf{r}'(t) = \left\langle 2te^{t^2}, \frac{3}{1+3t}\right\rangle$

(f) $\mathbf{r}'(t) = \left\langle \left[a\ t(-3\sin(3t)) + a\cos(3t)\right], b\cdot 3\sin^2 t\cos t, c\cdot 3\cos^2 t(-\sin t)\right\rangle$

$= \left\langle \left(a\cos(3t) - 3at\sin(3t)\right), 3b\sin^2 t\cos t, -3c\cos^2 t\sin t\right\rangle$

Problem code: VEZFS

8. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. Suppose $\mathbf{u}$,$\mathbf{v}$ are differentiable, c is a scalar, and f is a real-valued function. Prove that the derivative distributes over sums of functions:

$$\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$\begin{aligned}
\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] &= \frac{d}{dt}\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\rangle \\
&= \left\langle \frac{d}{dt}\left[u_1(t) + v_1(t)\right], \frac{d}{dt}\left[u_2(t) + v_2(t)\right], \frac{d}{dt}\left[u_3(t) + v_3(t)\right]\right\rangle \\
&= \langle u_1'(t) + v_1'(t), u_2'(t) + v_2'(t), u_3'(t) + v_3'(t)\rangle \\
&= \langle u_1'(t), u_2'(t), u_3'(t)\rangle + \langle v_1'(t), v_2'(t), v_3'(t)\rangle = \mathbf{u}'(t) + \mathbf{v}'(t)
\end{aligned}$$

Problem code: ZKKTG

9. Given the information from question 8, prove that the product rule applies in the expected way for vector-valued functions.

$$\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

$$\begin{aligned}\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] &= \frac{d}{dt}\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\rangle \\ &= \left\langle \frac{d}{dt}\left[f(t)u_1(t)\right], \frac{d}{dt}\left[f(t)u_2(t)\right], \frac{d}{dt}\left[f(t)u_3(t\right]\right\rangle \\ &= \langle f'(t)u_1(t) + f(t)u_1'(t), f'(t)u_2(t) + f(t)u_2'(t), f'(t)u_3(t) + f(t)u_3'(t)\rangle \\ &= f'(t)\langle u_1(t), u_2(t), u_3(t)\rangle + f(t)\langle u_1'(t), u_2'(t), u_3'(t)\rangle = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)\end{aligned}$$

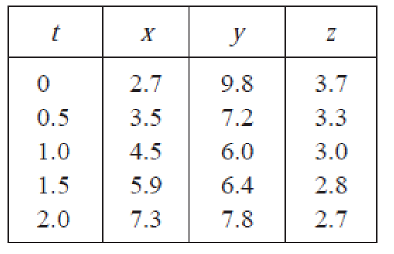
Problem code: PGAYY

Motion In Space: Velocity And Acceleration

10. The table below gives coordinates of a particle moving through space along a smooth curve.

    (a) Find the average velocities over the time intervals $[0, 1]$, $[0.5, 1]$, $[1, 2]$, and $[1, 1.5]$.

    (b) Estimate the velocity and speed of the particle at $t = 1$.



t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

(a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time t, then the average velocity over time interval [0, 1] is

$$v_{avg} = \frac{\mathbf{r}(1) + \mathbf{r}(0)}{1 - 0} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$$

Similarly, over the other intervals we have

$$\begin{aligned}[0.5, 1]: \quad v_{avg} &= \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k} \\ [1, 2]: \quad v_{avg} &= \frac{\mathbf{r}(12 - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0, \mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3, \mathbf{k} \\ [1, 1.5]: \quad v_{avg} &= \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}\end{aligned}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals [0.5, 1] and [1, 1.5], so

$$v(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$$

Then the speed is

$$||v(1)|| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$$

Problem code: CSPKZ

11. Find the velocity, acceleration, and speed of particles with the following position functions. Sketch the path of each particle and draw the velocity and acceleration vectors for the specified values of t.

(*a*) $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t\rangle, \quad t = 2$

(*b*) $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t}\rangle, \quad t = 1$

(*c*) $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \pi/3$

(*d*) $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad t = 0$

(*e*) $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}, \quad t = 1$

(*f*) $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

(a)

$$\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t\right\rangle \quad \Rightarrow$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -t, 1\rangle$$

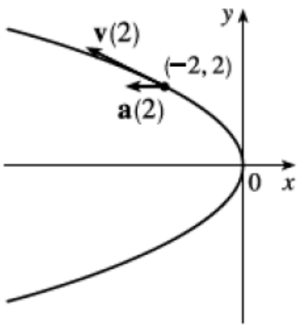
$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -1, 0\rangle$$

$$||\mathbf{v}(t)|| = \sqrt{t^2+1}$$

At $t = 2$:

$$\mathbf{v}(2) = \langle -2, 1\rangle$$

$$\mathbf{a}(2) = \langle -1, 0\rangle$$



(b)

$$\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t}\rangle \quad \Rightarrow$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -1, 2/\sqrt{t}\rangle$$

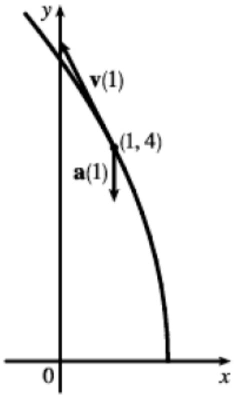
$$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 0, -1/t^{3/2}\right\rangle$$

$$||\mathbf{v}(t)|| = \sqrt{1 + 4/t}$$

At $t = 1$:

$$\mathbf{v}(1) = \langle -1, 2\rangle$$

$$\mathbf{a}(1) = \langle 0, -1\rangle$$



(c)

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} \quad \Rightarrow$$

$$\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

$$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$$

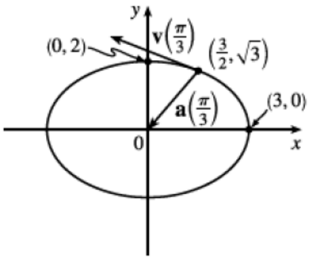
$$||\mathbf{v}(t)|| = \sqrt{9\sin^2 t + 4\cos^2 t} =$$

$$\sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t} = \sqrt{5\sin^2 t + 4}$$

At $t = \pi/3$:

$$\mathbf{v}(\frac{\pi}{3}) = -\frac{3\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j}$$

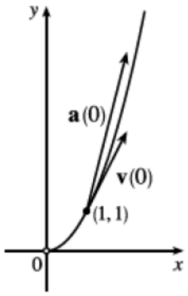
$$\mathbf{a}(\frac{\pi}{3}) = -\frac{3}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$$



(d)

$$\begin{aligned}\mathbf{r}(t) &= e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} \;\Rightarrow\\ \mathbf{v}(t) &= e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}\\ \mathbf{a}(t) &= e^t\,\mathbf{i} + 4e^{2t}\,\mathbf{j}\\ ||\mathbf{v}(t)|| &= \sqrt{e^{2t}+4e^{4t}} = e^t\sqrt{1+4e^{2t}}\end{aligned}$$

$$\begin{aligned}&\text{At } t=0:\\ &\mathbf{v}(0) = \mathbf{i} + 2\,\mathbf{j}\\ &\mathbf{a}(0) = \mathbf{i} + 4\,\mathbf{j}\end{aligned}$$

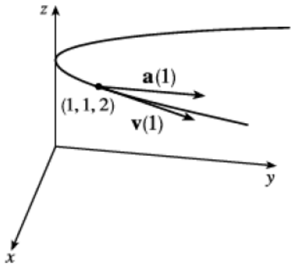


(e)

$$\begin{aligned}\mathbf{r}(t) &= t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k} \;\Rightarrow\\ \mathbf{v}(t) &= \mathbf{i} + 2t\,\mathbf{j}\\ \mathbf{a}(t) &= 2\,\mathbf{j}\\ ||\mathbf{v}(t)|| &= \sqrt{1+4t^2}\end{aligned}$$

$$\begin{aligned}&\text{At } t=1:\\ &\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}\\ &\mathbf{a}(1) = 2\,\mathbf{j}\end{aligned}$$

Here $x = t, y = t^2 \;\Rightarrow\; y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.

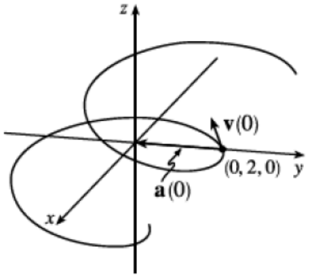


(f)

$$\begin{aligned}\mathbf{r}(t) &= t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k} \;\Rightarrow\\ \mathbf{v}(t) &= \mathbf{i} - 2\sin t\,\mathbf{j} + \cos t\,\mathbf{k}\\ \mathbf{a}(t) &= -2\cos t\,\mathbf{j} - \sin t\,\mathbf{k}\\ ||\mathbf{v}(t)|| &= \sqrt{1+4\sin^2 t+\cos^2 t} = \sqrt{2+3\sin^2 t}\end{aligned}$$

$$\begin{aligned}&\text{At } t=0:\\ &\mathbf{v}(0) = \mathbf{i} + \mathbf{k}\\ &\mathbf{a}(0) = -2\,\mathbf{j}\end{aligned}$$

Since $y^2/4 + z^2 = 1, x = t$, the path of the particle is an elliptical helix about the x-axis.



Problem code: BQZBM

12. Find the velocity, acceleration, and speed of particles with the given position functions.
    - *(a)* $\mathbf{r}(t) = \langle t^2+t, t^2-t, t^3\rangle$
    - *(b)* $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle$
    - *(c)* $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$
    - *(d)* $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$
    - *(e)* $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$
    - *(f)* $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle, \quad t \geq 0$

(a) $\mathbf{r}(t) = \left\langle t^2+t, t^2-t, t^3\right\rangle \;\Rightarrow\; \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t+1, 2t-1, 3t^2\right\rangle,\;\; \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 6t\rangle$

$||\mathbf{v}(t)|| = \sqrt{(2t+1)^2+(2t-1)^2+(3t^2)^2} = \sqrt{9t^4+8t^2+2}$

(b) $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle \;\Rightarrow\; \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t\rangle,\;\; \mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t\rangle$

$||\mathbf{v}(t)|| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}$

(c) $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-2}\,\mathbf{k} \;\Rightarrow\; \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k},\;\; \mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

$||\mathbf{v}(t)|| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}$

(d) $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k} \;\Rightarrow\; \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + 2\,\mathbf{j} + (1/t)\,\mathbf{k},\;\; \mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} - (1/t^2)\,\mathbf{k}$

$$||\mathbf{v}(t)|| = \sqrt{4t^2 + 4 + (1/t^2)} = \sqrt{[2t + (1/t)]^2} = \left|2t + \frac{1}{t}\right|$$

(e) $\mathbf{r}(t) = e^t\langle \cos t, \sin t, t\rangle \;\Rightarrow$

$$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\langle \cos t, \sin t, t\rangle + e^t\langle -\sin t, \cos t, 1\rangle = e^t\langle \cos t - \sin t, \sin t + \cos t, t + 1\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = e^t\langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1\rangle = e^t\langle -2\sin t, 2\cos t, t + 2\rangle$$

$$||\mathbf{v}(t)|| = e^t\sqrt{\cos^2 t + \sin^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1} = e^t\sqrt{t^2 + 2t + 3}$$

(f) $\mathbf{r}(t) = \left\langle t^2, \sin t - t\cos t, \cos t + t\sin t\right\rangle \;\Rightarrow$

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, \cos t - (-t\sin t + \cos t), -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, t\cos t + \sin t, -t\sin t + \cos t\rangle$$

$$||\mathbf{v}(t)|| = \sqrt{rt^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}\,t \quad [\text{since } t \geq 0].$$

Problem code: NJPTC

13. Define a vector-valued function for the path of a particle in $\mathbb{R}^2$ that:

- begins at $(x, y) = (-3, 1)$ at $t = 0$;
- moves clockwise around a circle of radius 6, centered at the point (3, 1),
- moves at a constant speed of 8 m/s. (Assume t is in seconds, and x and y are in meters.)

There are several possible answers to this question, though they are all based on similar constructions.

Recall that a basic counter-clockwise motion around the origin, with radius 1, is given by $\mathbf{r}(t) = \langle \cos(t), \sin(t)\rangle$. We can simply make incremental changes to this form to get the more complicated desired behaviour.

$$\text{Radius 6: } \mathbf{r}(t) = \langle 6\cos(t),\;\; 6\sin(t)\rangle$$
$$\text{Center at (3, 1): } \mathbf{r}(t) = \langle 6\cos(t) + 3,\;\; 6\sin(t) + 1\rangle$$

The other points require a little more explanation or check-and-correct.

Start at $(-3, 1)$

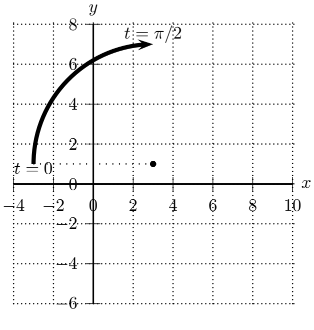
$\mathbf{r}(t) = \langle 6\cos(t) + 3,\;\; 6\sin(t) + 1\rangle$ starts at $\langle 6 + 3, 0 + 1\rangle = \langle 9, 1\rangle$. The start y is fine, but the start x needs to be changed to the opposite/left end of the circle. We can do that by *subtracting* the $6\cos(t)$ instead of adding it.

$\mathbf{r}(t) = \langle -6\cos(t) + 3,\;\; 6\sin(t) + 1\rangle \to \mathbf{r}(0) = \langle -6 + 3, 1\rangle = \langle -3, 1\rangle$

Clockwise rotation

We check to see which way the function is rotating now. One way is by subbing in some time values.

$$\mathbf{r}(t) = \langle -6\cos(t) + 3,\;\; 6\sin(t) + 1\rangle$$
$$\text{so } \mathbf{r}(0) = \langle -3, 1\rangle$$
$$\text{and } \mathbf{r}(\pi/2) = \langle 3, 7\rangle$$



This rotation is already clockwise, as seen in the sketch to the right.

Constant speed of 7 m/s

We change the speed by modifying the *multiplier of t inside the sine and cosine.*

$\mathbf{r}(t) = \langle -6\cos(At) + 3,\;\; 6\sin(At) + 1\rangle$ has velocity

$\mathbf{r}'(t) = \langle +6A\sin(At),\;\; 6A\cos(At)\rangle$ which gives a speed/magnitude of

$$\begin{aligned} \text{speed} = ||\mathbf{r}'(t)|| &= \sqrt{(6A\sin(At))^2 + (6A\cos(At))^2} \\ &= \sqrt{36A^2\sin^2(AT) + 36A^2\cos^2(AT)} \\ \text{common factors: } &= 6A\sqrt{\sin^2(AT) + \cos^2(AT)} \\ \text{trig identity: } &= 6A\sqrt{1} \\ &= 6A \end{aligned}$$

So to achieve a constant speed of 8 m/s, we need to pick $8 = 6A$, or $A = 8/6 = 4/3$.

Thus a working solution is $\mathbf{r}(t) = \langle -6\cos\left(\frac{4}{3}t\right) + 3,\ \ 6\sin\left(\frac{4}{3}t\right) + 1\rangle$

Problem code: YBWPC

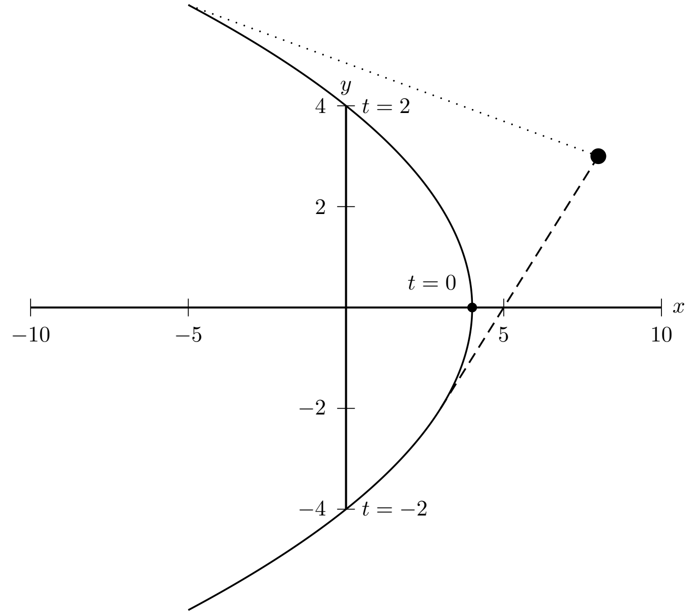
14. Your battle robot, equipped with a forward-facing cannon, is attempting to shoot a projectile at a target located at $(8, 3)$. To avoid your own robot being hit, you've programmed the robot to follow a somewhat complicated trajectory given by $\langle -t^2 + 4, 2t\rangle$, over the time interval $-10 \le t \le 10$.

    (a) Sketch the trajectory and the target point to estimate the possible launch times where the cannon could hit the target.

    (b) Determine the exact time value(s) that your cannon could fire and hit the target.

    Assume the robot, and its cannon, are always facing in the direction of the current velocity vector.

(a) To sketch the graph, you can sub in some test points from $t = 10$ to $t = 10$.



Based on the graph, it looks like there are two possible launch times where a tangent line would pass through $(8, 3)$:

- One time between $t = -2$ and $t = 0$, and
- a second time after $t = 2$.

Note from the directions though that **only the point before $t = 0$ can launch *forward* to hit the target**. The other tangent line (dotted in the diagram) would require the cannon to fire *backwards* to hit the target, and that isn't allowed in our scenario.

(b) To identify the launch times, we build a parametric formula for the tangent lines. Assume that we fire the cannon at time $t = a$, our position over time on the tangent line will be defined by

$$\mathbf{L}(t) = \underbrace{\mathbf{r}(a)}_{\text{start pos}} + \underbrace{\mathbf{v}(a)}_{\text{vel}} \cdot \underbrace{(t-a)}_{\text{time after launch}}$$

We have $\mathbf{r}(t) = \langle -t^2 + 4, 2t\rangle$.
Taking its derivative, we get the velocity
$\mathbf{v}(t) = \langle -2t, 2\rangle$.
Evaluating both at time $t = a$, we get the line formula $\mathbf{L}(t) = \langle -a^2 + 4, 2a\rangle + \langle -2a, 2\rangle \cdot (t - a)$.
We now use our target point to solve for possible a values: we require that $\mathbf{L}(t) = (8, 3)$ for some t value, or $(8, 3) = \langle -a^2 + 4, 2a\rangle + \langle -2a, 2\rangle \cdot t$.
Looking at the simpler y coordinates first, we require

$$\begin{aligned} 3 &= 2a + (2)(t - a) \\ \text{Solving for t:} \quad 3 - 2a &= 2(t - a) \\ \frac{3}{2} &= t \end{aligned}$$

Substitution that t value into the x coordinate equation,

$$\begin{aligned} 8 &= -a^2 + 4 + (-2a)(t - a) \\ 8 &= -a^2 + 4 + (-2a)\left(\frac{3}{2} - a\right) \\ 8 &= -a^2 + 4 + -3a + 2a^2 \\ 0 &= a^2 - 3a - 4 \\ 0 &= (a - 4)(a + 1) \end{aligned}$$

Our possible launch times are therefore $a = +4$ and $a = -1$ seconds.

From part (a) however, we know that only the launch time at $a = -1$ seconds will let the cannon shoot *forward* to hit the target, so our launch time should be at $t = -1$ seconds during the trajectory.

Problem code: JQVMR (Video Solution by K. A.)