Week #3 - Differentiation of Vector-Valued Functions

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Derivatives of Vector Functions

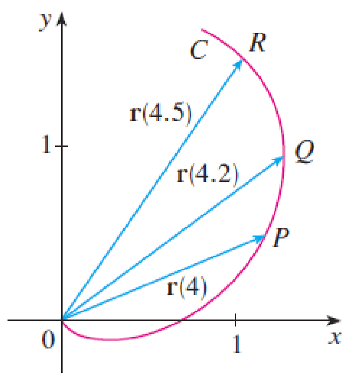
1. The figure shows a curve C given by a vector function $\mathbf{r}(t)$.

   (a) Draw the vectors $\mathbf{r}(4.5) - \mathbf{r}(4)$ and $\mathbf{r}(4.2) - \mathbf{r}(4)$.

   (b) Draw the vectors $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5}$ and $\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2}$

   (c) Write expressions for $\mathbf{r}'(t)$ and the unit tangent vector $\mathbf{T}(4)$.

   (d) Draw the vector $\mathbf{T}(4)$.



2. (a) Make a large sketch of the curve described by the vector function $\mathbf{r}(t) = \langle t^2, t\rangle$, $0 \le t \le 2$, and draw the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.

   (b) Draw the vector $\mathbf{r}'(1)$ starting at $(1, 1)$, and compare it with the vector $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$. Explain why these vectors are so close to each other in length and direction.

3. For the vector equation $\mathbf{r}(t) = \langle t - 2, t^2 + 1\rangle$

   (a) Sketch the plane curve.

   (b) Find $\mathbf{r}'(t)$.

   (c) Sketch the position vector $\mathbf{r}(t)$ and the velocity vector $\mathbf{r}'(t)$ for $t = -1$.

4. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $\quad t = 1$.

5. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle \sin t, 2\cos t\rangle$, $\quad t = \pi/4$.

6. Repeat question 3 for the vector equation $\mathbf{r}(t) = \langle e^t, e^{-t}\rangle$, $\quad t = 0$.

7. Find the derivative of the following vector functions.

   (a) $\mathbf{r}(t) = \langle t\sin(t), t^2, t\cos(2t)\rangle$.

   (b) $\mathbf{r}(t) = \langle \tan(t), \sec(t), 1/t^2\rangle$

   (c) $\mathbf{r}(t) = \langle t, 1, 2\sqrt{t}\rangle$

   (d) $\mathbf{r}(t) = \langle \dfrac{1}{1+t}, \dfrac{t}{1+t}, \dfrac{t^2}{1+t}\rangle$

   (e) $\mathbf{r}(t) = \langle e^{t^2}, -1, \ln(1 + 3t)\rangle$

   (f) $\mathbf{r}(t) = \langle a\ t\cos(3t), b\sin^3(t), c\cos^3 t\rangle$, where a, b and c are constants.

8. Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. Suppose $\mathbf{u}$,$\mathbf{v}$ are differentiable, c is a scalar, and f is a real-valued function. Prove that the derivative distributes over sums of functions:

$$\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

9. Given the information from question 8, prove that the product rule applies in the expected way for vector-valued functions.

$$\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

Motion In Space: Velocity And Acceleration

10. The table below gives coordinates of a particle moving through space along a smooth curve.

    (*a*) Find the average velocities over the time intervals $[0, 1]$, $[0.5, 1]$, $[1, 2]$, and $[1, 1.5]$.

    (*b*) Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

11. Find the velocity, acceleration, and speed of particles with the following position functions. Sketch the path of each particle and draw the velocity and acceleration vectors for the specified values of t.

    (*a*) $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t\rangle, \quad t = 2$

    (*b*) $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t}\rangle, \quad t = 1$

    (*c*) $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad t = \pi/3$

    (*d*) $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j}, \quad t = 0$

    (*e*) $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}, \quad t = 1$

    (*f*) $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad t = 0$

12. Find the velocity, acceleration, and speed of particles with the given position functions.

    (*a*) $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3\rangle$

    (*b*) $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle$

    (*c*) $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

    (*d*) $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$

    (*e*) $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k})$

    (*f*) $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle, \quad t \geq 0$

13. Define a vector-valued function for the path of a particle in $\mathbb{R}^2$ that:

    - begins at $(x, y) = (-3, 1)$ at $t = 0$;
    - moves clockwise around a circle of radius 6, centered at the point (3, 1),
    - moves at a constant speed of 8 m/s. (Assume t is in seconds, and x and y are in meters.)

14. Your battle robot, equipped with a forward-facing cannon, is attempting to shoot a projectile at a target located at $(8, 3)$. To avoid your own robot being hit, you've programmed the robot to follow a somewhat complicated trajectory given by $\langle -t^2 + 4, 2t \rangle$, over the time interval $-10 \le t \le 10$.

    (a) Sketch the trajectory and the target point to estimate the possible launch times where the cannon could hit the target.

    (b) Determine the exact time value(s) that your cannon could fire and hit the target.

    Assume the robot, and its cannon, are always facing in the direction of the current velocity vector.