Week #2 - Vector-Valued Functions

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Vector-Valued Functions

1. Find the domain of the vector function $\mathbf{r}(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$

The component functions $\sqrt{4-t^2}$, e^{-3t}, and $\ln(t+1)$ are all defined when:
input to square root is zero or greater $\Rightarrow 4 - t^2 \geq 0 \quad \Rightarrow \quad -2 \leq t \leq 2$, and
input to ln is greater than zero $\Rightarrow t + 1 > 0 \quad \Rightarrow \quad t > -1$,
so the domain of $\mathbf{r}(t)$ is $(-1, 2]$.

Problem code: RSJMJ (Video Solution by H.G.)

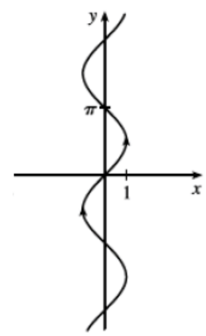
2. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \frac{t-2}{t+2},\ (\sin t),\ \ln(9-t^2) \right\rangle$

The component functions $\frac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t \neq -2$ and $9 - t^2 > 0 \quad \Rightarrow \quad -3 < t < 3$, so the domain of $\mathbf{r}$ is $(-3,-2) \cup (-2, 3)$.

Problem code: TNYXA (Video Solution by D. C.)

3. Sketch the curve of the vector equation $\mathbf{r}(t) = \langle \sin t, t \rangle$ and indicate with an arrow the direction in which t increases.

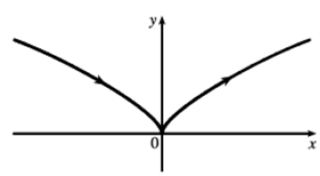
The corresponding parametric equations for this curve are $x = \sin t$, $y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \quad \Rightarrow \quad x = \sin y$, with $y \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated by the graph.



Problem code: DGHUZ (Video Solution by K.J.)

4. Sketch the curve of the vector equation $\mathbf{r}(t) = \langle t^3, t^2 \rangle$ and indicate with an arrow the direction in which t increases.

The corresponding parametric equations for this curve are $x = t^3$, $y = t^2$. We can make a table of values, or we can eliminate the parameter: $x = t^3 \quad \Rightarrow \quad t = \sqrt[3]{x} \quad \Rightarrow \quad y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t \in \mathbb{R} \quad \Rightarrow \quad x \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



Problem code: ZFDVT

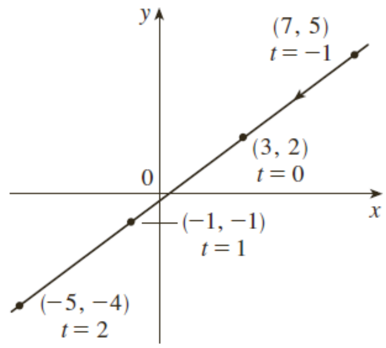
5. For $x = 3 - 4t, \quad y = 2 - 3t$

   a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

   b) Eliminate the parameter to find a Cartesian equation of the curve.

a) $x = 3 - 4t, \quad y = 2 - 3t$

t	-1	0	1	2
x	7	3	-1	-5
y	5	2	-1	-4



b) $x = 3 - 4t \;\Rightarrow\; 4t = -x + 3 \;\Rightarrow\; t = -\frac{1}{4}x + \frac{3}{4}$, so
$y = 2 - 3t = 2 - 3(-\frac{1}{4}x + \frac{3}{4}) = 2 + \frac{3}{4} - \frac{9}{4} \;\Rightarrow\; y = \frac{3}{4}x - \frac{1}{4}$

Problem code: YDYCC (Video Solution by K.M.)

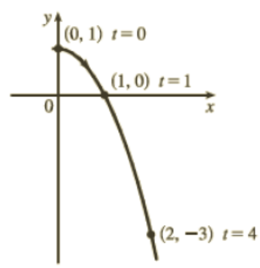
6. For $x = \sqrt{t},\ \ y = 1 - t$;

   a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

   b) Eliminate the parameter to find a Cartesian equation of the curve.

a) $x = \sqrt{t},\ \ y = 1 - t$

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



b) $x = \sqrt{t} \;\Rightarrow\; t = x^2 \;\Rightarrow\; y = 1 - t = 1 - x^2$. Since $t \geq 0, x \geq 0$. So the curve is the right half of the parabola $y = 1 - x^2$.

Problem code: MNDCQ

7. Consider the trajectory defined by $\mathbf{r}(t) = \langle 3\cos(3t), 2 + \cos^2(3t)\rangle$.

   (a) Eliminate the parameter from the trajectory to get the xy relationship in the trajectory.

   (b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

We start by separating this vector-valued function into the component functions:

$$x(t) = 3\cos(3t)$$
$$y(t) = 2 + \cos^2(3t)$$

Faced with trig functions, we would like to avoid solving down to the $t = \dots$ form, because that expression will involve inverse trig functions. (Eliminating the parameter *can* be done that way, it's just ugly and messy.)

As we look for alternatives, we note that both functions contain the cosine of t. Reformatting the $y(t)$ a little:

$$x = 3\underbrace{\cos(3t)}_{\text{common}}$$
$$y = 2 + (\ \underbrace{\cos(3t)}_{\text{common}}\ )^2$$

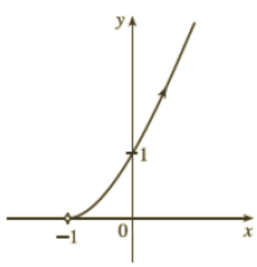
In other words, if we can isolate $(\cos(3t))$ in one equation, we can sub it into the other, and our t's will be eliminated.
Solving for $\cos(3t)$ in the x equation, we get
$\frac{x}{3} = \cos(3t)$.
Subbing that into our y equation, we get
$y = 2 + (\cos(3t))^2 = 2 + \left(\frac{x}{3}\right)^2 = 2 + \frac{x^2}{9}$.
We can then conclude that $y = 2 + \frac{x^2}{9}$ is a Cartesian equation for the trajectory of $\vec{r}(t)$.
Problem code: UPQLP (Video Solution by N.P.)

8. For $x = e^t - 1,\ \ y = e^{2t}$,

   a) Eliminate the parameter to find a Cartesian equation of the curve.
   b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

(a) We note that e^t is shared in common by both the x and y components. From the x component we immediately get that $e^t = x$.
Subbing that into the y equation we get:
$y = (e^t)^2 = (x+1)^2$.

(b) Looking at the range of possible x values, we notes that exponentials like e^t are always positive, so $x = e^t - 1$ must always be greater than -1. Since $x > -1$, the actual trajectory of this curve is just the right side of the parabola $y = (x+1)^2$.



Problem code: ZZUBZ

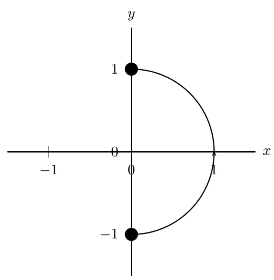
9. For $x = \cos\left(\frac{1}{2}\theta\right)$, $y = \sin\left(\frac{1}{2}\theta\right)$ and $-\pi \leq \theta \leq \pi$,

   (a) Eliminate the parameter to find a Cartesian equation of the curve.
   (b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

(a) We note that we have a sine/cosine pair, which would represent a circular motion. We can confirm this by squaring both components and adding them:

$$x^2 + y^2 = \left(\cos\left(\frac{1}{2}\theta\right)\right)^2 + \left(\sin\left(\frac{1}{2}\theta\right)\right)^2$$
$$= 1$$

(b) Looking at the range of possible θ values, we would start at $\theta = -\pi \rightarrow x = \cos(-\pi/2) = 0$ and $y = \sin(-\pi/2) = -1$: (0, -1).
At the end of the interval, $\theta = \pi \rightarrow x = \cos(\pi/2) = 0$ and $y = \sin(\pi/2) = 1$: (0, 1).
In the middle (to see the direction of travel), $\theta = 0 \rightarrow x = \cos(0) = 1$ and $y = \sin(0) = 0$: (1, 0).
The path traced out by the given functions is shown below.



Problem code: OWIES

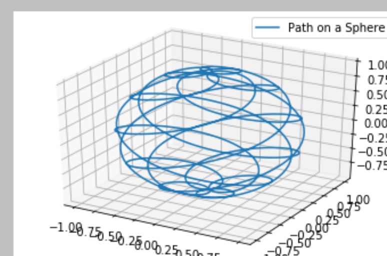
10. Define a vector-valued function that traces out an ellipse centered at the origin, with an x-axis 'radius' of 7 and a y-axis 'radius' of 3.

Multiple answers are possible, but all will be based on a sine/cosine pair, and have a 7 multiplier on the x side and a 3 on the y side. E.g. all of the following would satisfy the criteria given:

- $\mathbf{r}(t) = \langle 7\cos(t), 3\sin(t)\rangle$
- $\mathbf{r}(t) = \langle 7\sin(t), 3\cos(t)\rangle$
- $\mathbf{r}(t) = \langle -7\cos(t), -3\sin(t)\rangle$

Problem code: VQQUZ

11. Show that the trajectory defined by
$\mathbf{u}(t) = \langle \sin(t)\cos(2.75t), \sin(t)\sin(2.75t), \cos(t)\rangle$
lies on the unit sphere, defined by $x^2 + y^2 + z^2 = 1$.



For this problem, we need to take the given x, y and z coordinates of the trajectory $\mathbf{u}(t)$, and show that they satisfy the equation given for the sphere.

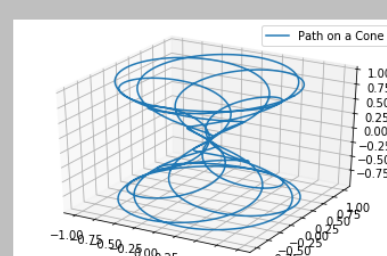
Starting at the LHS of the sphere equation,

$$\begin{aligned} & x^2 + y^2 + z^2 \\ & = [\sin(t)\cos(2.75t)]^2 + [\sin(t)\sin(2.75t)]^2 + [\cos(t)]^2 \\ & = \sin^2(t)\cos^2(2.75t) + \sin^2(t)\sin^2(2.75t) + \cos^2(t) \\ \text{Factoring the first terms: } & = \sin^2(t)[\cos^2(2.75t) + \sin^2(2.75t)] + \cos^2(t) \\ \text{Applying the } \cos^2(2.75t) + \sin^2(2.75t) = 1 \text{ identity: } & = \sin^2(t)[1] + \cos^2(t) \\ \text{And the same identity again, but with } \sin^2(t), \cos^2(t)\text{: } & = 1 \end{aligned}$$

which is the RHS of the unit sphere equation, $x^2 + y^2 + z^2 = 1$. Thus all the points on the trajectory $\mathbf{u}(t)$ lie on the unit sphere.

Problem code: PUKML

12. Show that the trajectory defined by
$\mathbf{u}(t) = \langle \cos(t)\cos(2.75t), \cos(t)\sin(2.75t), \cos(t)\rangle$
lies on the cone defined by $x^2 + y^2 = z^2$.



For this problem, we need to take the given x, y and z coordinates of the trajectory $\mathbf{u}(t)$, and show that they satisfy the equation given for the cone.

Starting at the LHS of the cone equation,

$$\begin{aligned}
& x^2 + y^2 \\
& = [\cos(t)\cos(2.75t)]^2 + [\cos(t)\sin(2.75t)]^2 \\
& = \cos^2(t)\cos^2(2.75t) + \cos^2(t)\sin^2(2.75t) \\
\text{Factoring the first terms:} \quad & = \cos^2(t)[\cos^2(2.75t) + \sin^2(2.75t)] \\
\text{Applying the } \cos^2(2.75t) + \sin^2(2.75t) = 1 \text{ identity:} \quad & = \cos^2(t)[1] \\
\text{but } z(t) = \cos(t)\text{, so this equals:} \quad & = z^2
\end{aligned}$$

which is the RHS of the cone equation, $x^2 + y^2 = z^2$. Thus all the points on the trajectory $\mathbf{u}(t)$ lie on the cone.

Problem code: RMBSU (Video Solution by D. C.)

13. Describe the motion of a particle with position (x, y) as t varies, given by $x = \sin\ t$, $y = \cos^2\ t$, $-2\pi \leq t \leq 2\pi$

$y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point (0, 1), moves to (1, 0), and goes back to (0, 1). As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to (0, 1). The particle repeats this motion as t goes from 0 to 2π.

Problem code: UEYFV

14. Suppose the trajectories of two particles are given by the vector functions

$$\mathrm{r}_1(t) = \langle t^2, 7t - 12, t^2 \rangle \qquad \mathrm{r}_2(t) = \langle 4t - 3, t^2, 5t - 6 \rangle$$

for $t \geq 0$. Do the particles collide?

For the particles to collide, we require $\mathrm{r}_1(t) = \mathrm{r}_2(t) \Leftrightarrow \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives

$$\begin{aligned}
t^2 &= 4t - 3, \\
7t - 12 &= t^2, \text{ and} \\
t^2 &= 5t - 6.
\end{aligned}$$

From the first equation, $t^2 - 4t + 3 = 0 \Leftrightarrow (t - 3)(t - 1) = 0$, so $t = 1$ or $t = 3$.
$t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when t = 3, and the point (9, 9, 9).

Problem code: UVCMV

15. Two particles travel along the space curves

$$r_1(t) = \langle t^3, t, t^2 \rangle \qquad r_2(t) = \langle 1 + 26t, 1 + 2t, 1 + 8t \rangle$$

Do the particles collide? Do their paths intersect?

The particles *collide* provided $\mathrm{r}_1(t) = \mathrm{r}_2(t) \Leftrightarrow \langle t^3, t, t^2 \rangle = \langle 1 + 26t, 1 + 2t, 1 + 8t \rangle$. Equating components gives

$$\begin{aligned}
t^3 &= 1 + 26t, \\
t &= 1 + 2t, \text{ and} \\
t^2 &= 1 + 8t.
\end{aligned}$$

The second equation gives $t = -1$, but this does not satisfy the other equations, so the particles do **not collide**.

For the paths to *intersect*, we need to find a value for t and a value for s where $\mathrm{r}_1(t) = \mathrm{r}_2(s) \Leftrightarrow \langle t^3, t, t^2 \rangle = \langle 1 + 26s, 1 + 2s, 1 + 8s \rangle$. Equating components,

$$t^3 = 1 + 26s, \tag{1}$$
$$t = 1 + 2s, \text{ and} \tag{2}$$
$$t^2 = 1 + 8s. \tag{3}$$

Substituting equation (2) into equation (3) gives

$$\begin{aligned} (1+2s)^2 &= 1+8s \\ \text{expanding: } 1+4s+4s^2 &= 1+8s \\ \text{moving all to the left: } 4s^2-4s &= 0 \\ \text{factoring : } 4s(s-1) &= 0 \end{aligned}$$

This gives possible solutions of

- $s = 0$, with matching $t = 1 + 2s = 1$, and
- $s = 1$, and matching $t = 1 + 2s = 3$.

Last, we need to check that equation (1) is also satisfied by both answers:

- $s = 0, t = 1$: LHS (1) $= t^3 = 1^3 = 1$. And also RHS (1) $= 1 + 26s = 1$.
- $s = 1, t = 3$: LHS (1) $= t^3 = 3^3 = 27$. And also RHS (1) $= 1 + 26s = 1 + 26 = 27$.

Thus the paths intersect twice,

- at the point (1, 1, 1) when $s = 0$ and $t = 1$, and
- at (27, 3, 9) when $s = 1$ and $t = 3$.

Problem code: QJSSS (Video Solution by K. A.)