Week #1 - Derivatives and Vectors

Some problems and solutions selected or adapted from Stewart Calculus and Hughes-Hallett Calculus-Early Transcendentals.

Computing Derivatives

Below are a **small sample of problems** involving the computation of derivatives. They are **not enough** to properly learn and memorize how to apply all the derivative rules. You should practice with as many problems as you need to become proficient at computing derivatives.

Further practice problems can be found in any calculus textbook.

1. Let $f(x) = 4e^x - 9x^2 + 5$. Compute $f'(x)$.

$f'(x) = 4e^x - 18x$

Problem code: ZEFKF (Video Solution by I.G.)

2. Let $f(x) = 2x^6\sqrt{x} + \dfrac{-5}{x^3\sqrt{x}}$. Compute $f'(x)$.

You can use the product rule here if you like, but it is far easier to rewrite the function before you start, since all the terms are just powers of x.

$$f(x) = 2x^6\sqrt{x} + \frac{-5}{x^3\sqrt{x}}$$

$$\begin{aligned} f(x) &= 2x^6 \cdot x^{1/2} + \frac{-5}{x^3 \cdot x^{1/2}} \\ &= 2x^{\frac{13}{2}} - 5x^{\frac{-7}{2}} \end{aligned}$$

Differentiating,

$$\begin{aligned} f'(x) &= 2\frac{13}{2}x^{\frac{11}{2}} - 5\left(\frac{-7}{2}\right)x^{\frac{-9}{2}} \\ &= 13x^5\sqrt{x} + \frac{35}{2x^4\sqrt{x}} \end{aligned}$$

Problem code: PVLBY (Video Solution by K.J.)

3. Let $f(x) = \dfrac{7x^2 + 7x + 5}{\sqrt{x}}$.
   (a) Compute $f'(x)$. (b) Find $f'(3)$.

You can use the quotient rule here if you like, but it is far easier to rewrite the function before you start, since all the terms are just powers of x.

$$\begin{aligned} f(x) &= \frac{7x^2 + 7x + 5}{\sqrt{x}} \\ &= 7x^{3/2} + 7x^{1/2} + 5x^{-1/2} \end{aligned}$$

(a) $f'(x) = \dfrac{21}{2}x^{1/2} + \dfrac{7}{2}x^{-1/2} - \dfrac{5}{2}x^{-3/2}$

(b) Evaluating $f'(x)$ at $x = 3$, we obtain ≈ 19.726.

Problem code: VMJJR

4. Let $f(t) = 7t^{-7}$.
   (a) Compute $f'(t)$. (b) Find $f'(3)$.

(a) $f'(x) = -49\ t^{-8}$

(b) $f'(3) = -49\ (3^{-8})$

Problem code: SVXQX (Video Solution by D. C.)

5. Let $f(x) = 4e^x + e^1$. Compute $f'(x)$.

Don't be thrown off by the e^1: that's a constant (equal to e or ≈ 2.7), so the derivative of that term is zero.

$f'(x) = 4e^x$

Problem code: YEQZE

6. Let $f(x) = 4e^x + 4x$. Compute $f'(x)$.

$f'(x) = 4e^x + 4$

Problem code: RLHXS (Video Solution by D. C.)

7. $f(x) = (3x^2 - 2)(6x + 3)$.
   (a) Compute $f'(x)$. (b) Find $f'(4)$.

You can either expand the product before differentiating (and obtain $f(x) = 18x^3 + 9x^2 - 12x - 6$), or use the product rule. Both give the same answer.

(a) $f'(x) = 54x^2 + 18x - 12$.

(b) $f'(4) = 924$.

Problem code: DVHWT (Video Solution by I.G.)

8. Let $f(x) = \dfrac{\sqrt{x} - 4}{\sqrt{x} + 4}$. Compute $f'(9)$.

We need to use the quotient rule for this function.

$$f'(x) = \frac{\frac{1}{2}x^{-1/2}(x^{1/2}+4) - (x^{1/2}-4)\left(\frac{1}{2}\right)x^{-1/2}}{(\sqrt{x}+4)^2}$$

factor out 1/2 from top:

$$= \frac{1}{2}\left(\frac{x^{-1/2}(x^{1/2}+4) - (x^{1/2}-4)x^{-1/2}}{(\sqrt{x}+4)^2}\right)$$

expanding the numerator terms,

noting $x^{-1/2} \cdot x^{1/2} = 1$, and $x^{-1/2} = /\sqrt{x}$:

$$= \frac{1}{2}\left(\frac{1 + (4/\sqrt{x}) - 1 + (4/\sqrt{x})}{(\sqrt{x}+4)^2}\right)$$

$$= \frac{1}{2}\left(\frac{8}{\sqrt{x}}\frac{1}{(\sqrt{x}+4)^2}\right)$$

$$= \frac{4}{\sqrt{x}(\sqrt{x}+4)^2}$$

$$f'(9) = \frac{4}{3(3+4)^2}$$

$$= \frac{4}{147}$$

Problem code: WJFVA

9. Consider $f(x) = \dfrac{4x+3}{3x+2}$.
   (a) Compute $f'(x)$. (b) Find $f'(5)$.

(a) $f'(x) = \dfrac{4(3x+2) - (4x+3)(3)}{(3x+2)^2}$

(b) $f'(5) = \dfrac{-1}{17^2}$

Problem code: CNQAE

10. Consider $f(x) = \dfrac{7-x^2}{7+x^2}$
   (a) Compute $f'(x)$. (b) Find $f'(1)$.

(a) $f'(x) = \dfrac{-28x}{(x^2+7)^2}$

(b) $f'(1) = \dfrac{-28}{64}$

Problem code: KTNUL

11. Let $f(x) = -2x(x-3)$.
   (a) Compute $f'(x)$. (b) Find $f'(-5)$.

(a) $f'(x) = -4x + 6$

(b) $f'(-5) = 26$

Problem code: FNYDT

12. $f(x) = \dfrac{4x^3-3}{x^4}$
   (a) Compute $f'(x)$. (b) Find $f'(2)$.

This function is easier to differentiated if we simplify it by breaking it into separate terms, then writing it in terms of powers.

$f(x) = \dfrac{4x^3-3}{x^4} = \dfrac{4}{x} - \dfrac{3}{x^4} = 4x^{-1} - 3x^{-4}$

(a) $f'(x) = -4x^{-2} + 12x^{-5}$

(b) $f'(2) = -0.625$

Problem code: WTAQH

13. $g(x) = \dfrac{e^x}{5+4x}$. Compute $g'(x)$.

No simplification of the function is available, so we use quotient rule:

$$g'(x) = \frac{(e^x)(5+4x) - (e^x)(4)}{(5+4x)^2}$$

$$= \frac{(e^x)[5+4x-4]}{(5+4x)^2}$$

$$= \frac{(1+4x)e^x}{(5+4x)^2}$$

Problem code: KSWRA

14. $f(x) = \dfrac{4x^2 \tan x}{\sec x}$.
   (a) Find $f'(x)$. (b) Find $f'(3)$.

This should definitely be simplified before you differentiate!

Recall: $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, and $\sec(x) = \dfrac{1}{\cos(x)}$, so $\dfrac{\tan(x)}{\sec(x)} = \dfrac{\sin(x)}{\cos(x)}\cos(x) = \sin(x)$.

This means that our original function can be written as

$f(x) = 4x^2 \sin(x)$.

(a) $f'(x) = 8x\sin(x) + 4x^2\cos(x)$

(b) $f'(3) = 24\sin(3) + 36\cos(3) \approx -32.253$

Problem code: AZGVM

15. $f(x) = 7\sin x + 12\cos x$
   (a)Compute $f'(x)$. (b) Find $f'(1)$.

(a) $f'(x) = 7\cos(x) - 12\sin(x)$

(b) $f'(1) = 7\cos(1) - 12\sin(1) \approx -6.3155$

Problem code: PSCJS

16. Let $f(x) = \cos x - 2\tan x$. Compute $f'(x)$.

$f'(x) = -\sin(x) - 2\sec^2(x)$

Problem code: JGJXG

17. $f(x) = \dfrac{5\sin x}{3+\cos x}$

(a) Compute $f'(x)$. (b) Find $f'(2)$.

Don't forget the identity $\sin^2(x) + \cos^2(x) = 1$.

(a) $f'(x) = (15\cos(x) + 5)/(3 + \cos(x))^2$

(b) $f'(2) = (15\cos(2) + 5)/(3 + \cos(2))^2 \approx -0.18606$

Problem code: ERSYB

18. $f(x) = 7x(\sin x + \cos x)$

(a) Compute $f'(x)$. (a) Find $f'(3)$.

(a) $f'(x) = 7(\sin(x) + \cos(x)) + 7x(\cos(x) - \sin(x))$

(b) $f'(3) \approx -29.695$

Problem code: HVKVV

19. Let $f(x) = \cos(\sin(x^2))$. Compute $f'(x)$.

$f'(x) = -\sin(\sin(x^2))\cos(x^2)\cdot(2x)$

Problem code: SXSEX

20. Let $f(x) = 2\sin^3 x$. Compute $f'(x)$.

Writing as nested powers, so easier to apply differentiation rules.
$f(x) = 2\sin^3 x = 2[\sin(x)]^3$.

$$\text{so } f'(x) = 2\cdot 3[\sin(x)]^2\cos(x) = 6\sin^2(x)\cos(x)$$

Problem code: QUBDL (Video Solution by K.J.)

21. Let $y = (8 + \cos^2 x)^6$. Compute $\dfrac{dy}{dx}$.

$\dfrac{dy}{dx} = 6(8 + \cos^2(x))^5(2\cos(x)(-\sin(x)))$

Problem code: LGZAK (Video Solution by C.C.)

22. Let $f(x) = -3\ln[\sin(x)]$. Compute $f'(x)$.

$f'(x) = \dfrac{-3}{\sin(x)}\cos(x) = -3\cot(x)$

Problem code: NZYJV

23. Let $f(x) = 2\ln(4 + x)$. Compute $f'(x)$.

$f'(x) = \dfrac{2}{4+x}$

Problem code: XHGKA (Video Solution by C.C.)

24. Let
$$P = \frac{V^2R}{(R+r)^2}.$$
Calculate $\dfrac{dP}{dr}$, assuming that r is variable and R and V are constant.

Note that V is also constant. Let
$$f(r) = \frac{V^2R}{(R+r)^2} = \frac{V^2R}{R^2+2Rr+r^2}.$$

Using the quotient rule:
$$f'(r) = \frac{(R^2+2Rr+r^2)(0) - (V^2R)(2R+2r)}{(R+r)^4} = -\frac{2V^2R(R+r)}{(R+r)^4} = -\frac{2V^2R}{(R+r)^3}.$$

Problem code: UDNUM

Vector Basics

For Problems 25-29, say whether the given quantity could be represented only as a vector, or if a scalar could be used to represent it.

25. The current population of Canada.

Scalar. It is a single value.

Problem code: KRDVQ

26. The distance from Toronto to Vancouver.

Scalar. We are asked for the distance, not the different components of the travel.

Problem code: GSHVE

27. The magnetic field at a point on the earth's surface.

A magnetic field has both a strength and a direction, so it would be a vector quantity.

Problem code: GTZXF

28. The current temperature at a point on the earth's surface.

Temperature is measured by a single number, so it is a scalar.

Problem code: GAFVR

29. The populations of each of the 13 provinces and territories.

Since we are measuring all 13 values simultaneously, and you need all 13 distinct numbers to represent the province-by-province information. Thus we would need a vector representation. You could also say that no single value could represent the populations of all 13 provinces and territories, so it can *not* be a scalar quantity.

Problem code: RBDUL

For Problems 30-37, find the resulting vector, and write it as both () a sum of unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, as well as (b) component-only form, $\langle a, b, c\rangle$.

30. $-4\langle 1, -2, 0\rangle - 0.5\langle 1, 0, -1\rangle$

$= \langle(-4 - 0.5), (-4)(-2), (-0.5)(-1)\rangle$

$= \langle -4.5, 8, 0.5\rangle$ or $= -4.5\vec{i} + 8\vec{j} + 0.5\vec{k}$

Problem code: AQZNY

31. $2\langle 0.45, -0.9, -0.01\rangle - 0.5\langle 1.2, 0, -0.1\rangle$

$= \langle 2(0.45) - -0.5(1.2), 2(-0.9), 2(-0.01) + (-0.5)(-0.1)\rangle$

$= \langle 0.3, -1.8, 0.03\rangle$ or $= 0.3\vec{i} - 1.8\vec{j} + 0.03\vec{k}$

Problem code: GTGBZ

32. $(3\vec{i} - 4\vec{j} + 2\vec{k}) - (6\vec{i} + 8\vec{j} - \vec{k})$

$$= -3\vec{i} - 12\vec{j} + 3\vec{k}$$
$$\text{or} = \langle -3, -12, 3\rangle$$

Problem code: JJDZD

33. $(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j})$

$\vec{i} + 3\vec{j} = \langle 1, 3\rangle$

Problem code: CBEKM

34. $(\vec{i} + 2\vec{j}) + (-3)(2\vec{i} + \vec{j})$

$= -5\vec{i} - \vec{j} = \langle -5, -1\rangle$

Problem code: TVGCB

35. $-4(\vec{i} - 2\vec{j}) - 0.5(\vec{i} - \vec{k})$

$-4.5\vec{i} + 8\vec{j} + 0.5\vec{k} = \langle -4.5, 8, 0.5\rangle$

Problem code: WGFKL

36. $2(0.45\vec{i} - 0.9\vec{j} - 0.01\vec{k}) - 0.5(1.2\vec{i} - 0.1\vec{k})$

$(0.9\vec{i} - 1.8\vec{j} - 0.02\vec{k}) - (0.6\vec{i} - 0.05\vec{k}) = 0.3\vec{i} - 1.8\vec{j} + 0.03\vec{k}$
$= \langle 0.3, -1.8, 0.03\rangle$

Problem code: JSAEA

37. $(4\vec{i} - 3\vec{j} + 7\vec{k}) - 2(5\vec{i} + \vec{j} - 2\vec{k})$

$$= (4 - 10)\vec{i} + (-3 - 2)\vec{j} + (7 + 4)\vec{k}$$
$$= -6\vec{i} - 5\vec{j} + 11\vec{k}$$
$$= \langle -6, -5, 11\rangle$$

Problem code: UHCRT

For Problems 38-41, perform the indicated operations on the following vectors:

$$\vec{a} = \langle 0, 2, 1\rangle, \quad \vec{b} = \langle 3, 5, 4\rangle$$
$$\vec{c} = \langle 1, 6, 0\rangle, \quad \vec{x} = \langle 2, 9, 0\rangle$$
$$\vec{y} = \langle 4, -7, 0\rangle, \quad \vec{z} = \langle 1, -3, -1\rangle$$

38. Find $\vec{a} + \vec{z}$.

$\vec{a} + \vec{z} = \langle 0, 2, 1\rangle + \langle 1, -3, -1\rangle$
$= \langle 1, -1, 0\rangle$

Problem code: DWAXU

39. Find $2\vec{c} + \vec{x}$.

$$2\vec{c} + \vec{x} = 2\langle 1, 6, 0\rangle + \langle 2, 9, 0\rangle$$
$$= \langle 4, 21, 0\rangle$$

Problem code: ZNLFZ

40. Find $2\vec{a} + 7\vec{b} - 5\vec{z}$.

$$2\langle 0, 2, 1\rangle + 7\langle 3, 5, 4\rangle - 5\langle 1, -3, -1\rangle$$
$$= \langle 0 + 21 - 5, 4 + 35 + 15, 2 + 28 + 5\rangle$$
$$= \langle 16, 54, 35\rangle$$

Problem code: WYKAF

41. Find $||\vec{y} - \vec{x}||$.

$||\langle 4, -7, 0\rangle - \langle 2, 9, 0\rangle|| = ||\langle 2, -16, 0\rangle|| = \sqrt{2^2 + (-16)^2 + 0^2} = \sqrt{260} \approx 16.12$

Problem code: CYBMW

42. Find a vector with length 2 that points in the same direction as $\vec{i} - \vec{j} + 2\vec{k}$.

We start with the vector $\vec{i} - \vec{j} + 2\vec{k}$. In components, this is $\langle 1, -1, 2\rangle$.

We can easily make a unit vector out of the original by dividing by its length, $L = \sqrt{1^2 + (-1)^2 + (2)^2} = \sqrt{6}$, meaning

$\vec{v} = \frac{1}{\sqrt{6}}\langle 1, -1, 2\rangle$ is a unit vector parallel to the original

If we want a vector in the same direction, but of length 2, we can take our unit vector and multiply it by 2:

$$\vec{w} = 2\vec{v} = 2\left(\frac{1}{\sqrt{6}}\langle 1, -1, 2\rangle\right) = \frac{2}{\sqrt{6}}\langle 1, -1, 2\rangle$$

or equivalently, but less aesthetically,

$$\vec{w} = \left\langle \frac{2}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{4}{\sqrt{6}} \right\rangle$$

Problem code: YEHFM

For Problems 43-46, consider the following scenario. A cat on the ground at the point (1, 4, 0) watches a squirrel at the top of a tree. The tree is one unit high with its base at (2, 4, 0). Find the displacement vectors for the points described in Problems 43-46.

43. From the origin to the cat.

$\vec{i} + 4\vec{j} + 0\vec{k}$ or $\langle 1, 4, 0\rangle$.

Problem code: LSJGA

44. From the bottom of the tree to the squirrel.

The squirrel is at the top of the tree, one unit above the base.

$1\vec{k}$ or $\langle 0, 0, 1\rangle$.

Problem code: QLFEW

45. From the bottom of the tree to the cat.

From (2, 4, 0) to (1, 4, 0): $-\vec{i}$ or $\langle -1, 0, 0\rangle$.

Problem code: CDHWB

46. From the cat to the squirrel.

From (1, 4, 0) to (2, 4, 1): $1\vec{i} + 0\vec{j} + 1\vec{k}$ or $\langle 1, 0, 1\rangle$.

Problem code: VTMSV

For Problems 47-51, find the length of the vectors given.

47. $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$

length $= \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

Problem code: EPGUH (Video Solution by L. K.)

48. $\vec{v} = \vec{i} - 3\vec{j} - \vec{k}$

length $= \sqrt{1^2 + (-3)^2 + (-1)^2} = \sqrt{11}$

Problem code: LHRGV (Video Solution by L. K.)

49. $\vec{v} = \langle 1, -1, 3\rangle$

length $= \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}$

Problem code: TFFWE (Video Solution by L. K.)

50. $\vec{v} = 7.2\vec{i} - 1.5\vec{j} + 2.1\vec{k}$

length $= \sqrt{(7.2)^2 + (-1.5)^2 + (2.1)^2} \approx 7.65$

Problem code: HDDJM (Video Solution by L. K.)

51. $\vec{v} = \langle 1.2, -3.6, 4.1\rangle$

length $= \sqrt{(1.2)^2 + (-3.6)^2 + (4.1)^2} \approx 5.58$

Problem code: SKDDF (Video Solution by L. K.)

52. For each of the four statements below, answer the following questions: Does the statement make sense? If yes, is it true for all possible choices of $\vec{a}$ and $\vec{b}$? If no, why not?

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(b) $\vec{a} + ||\vec{b}|| = ||\vec{a} + \vec{b}||$

(c) $||\vec{b} + \vec{a}|| = ||\vec{a} + \vec{b}||$

(d) $||\vec{a} + \vec{b}|| = ||\vec{a}|| + ||\vec{b}||$

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ is true for all choices of $\vec{a}$ and $\vec{b}$. Addition of vectors is commutative (order can be changed without changing the final value).

(b) $\vec{a} + ||\vec{b}|| \neq ||\vec{a} + \vec{b}||$. The left side of the equation makes no mathematical sense: you cannot add a vector and a scalar together.

(c) $||\vec{b} + \vec{a}|| = ||\vec{a} + \vec{b}||$ is true for all choices of $\vec{a}$ and $\vec{b}$. Since the vector sum is equal on both sides (part (a)), the resulting vectors are the same and so their lengths are also the same.

(d) $||\vec{a}+\vec{b}|| \neq ||\vec{a}|| + ||\vec{b}||$ for all choices of $\vec{a}$ and $\vec{b}$ (though is *is* true for *some* choices). To show that the sums are not necessarily equal, take $\vec{b} = -\vec{a}$. Then $||\vec{a}+\vec{b}|| = ||\vec{a}+(-\vec{a})|| = ||\vec{0}|| = 0$.
Looking at the other side of the equation, taking the lengths before we add gives $||\vec{a}|| + ||\vec{b}|| = ||\vec{a}|| + ||-\vec{a}|| = 2||\vec{a}||$, since the length of $\vec{a}$ and $-\vec{a}$ are the same. This does not equal the 0 value on the left side, so $||\vec{a}+\vec{b}|| \neq ||\vec{a}|| + ||\vec{b}||$ is not necessarily true.

Problem code: NSUZC

53. For what values of t are the following pairs of vectors parallel?

(a) $2\vec{i} + (t^2 + (2/3)t + 1)\vec{j} + t\vec{k}, 6\vec{i} + 8\vec{j} + 3\vec{k}$

(b) $t\vec{i} + \vec{j} + (t-1)\vec{k}, 2\vec{i} - 4\vec{j} + \vec{k}$

(c) $2t\vec{i} + t\vec{j} + t\vec{k}, 6\vec{i} + 3\vec{j} + 3\vec{k}$

(a) We need $\langle 6,8,3\rangle = \lambda\langle 2,(t^2+\frac{2}{3}t+1),t\rangle$ for some λ. I.e. the two vectors must be multiples of each other. Equating each component separately gives:

$$\begin{aligned} x \text{ coord:} \quad & 6 = 2\lambda \\ y \text{ coord:} \quad & 8 = (t^2 + (2/3)t + 1)\lambda \\ z \text{ coord:} \quad & 3 = t\lambda \end{aligned}$$

From the first equation, we have $\lambda = 3$. Substituting $\lambda = 3$ into the third equation gives $t = 1$. Check the second equation, it says $8 = 8$, if t = 1 and $\lambda = 3$. So for $t = 1$, the two vectors are parallel to each other.

(b) Similar to part (a), we need to solve

$$\begin{aligned} x \text{ coord:} \quad & t = 2\cdot\lambda \\ y \text{ coord:} \quad & 1 = -4\cdot\lambda \\ z \text{ coord:} \quad & (t-1) = 1\cdot\lambda \end{aligned}$$

From the first two equations we have $\lambda = -1/4$ and $t = -1/2$. Substituting this into the third equation gives $\frac{-3}{2} = \frac{-1}{4}$, which isn't true. Thus this system of equations has no solution, so the pair of vectors is not parallel to each other for any value of t.

(c) $2t\vec{i} + t\vec{j} + t\vec{k} = \dfrac{t}{3}(6\vec{i} + 3\vec{j} + 3\vec{k})$. Since, for any t, these two vectors are multiples of each other, they are always parallel regardless of the value of t.

Problem code: GWWXY

54. Find all vectors $\vec{v}$ in 2 dimensions having $||\vec{v}|| = 5$ and for which the $\vec{i}$-component of $\vec{v}$ is $3\vec{i}$.

Let $\vec{v} = a\vec{i} + b\vec{j}$. We are told that $a = 3$, so $\vec{v} = 3\vec{i} + b\vec{j}$. To have the length of the vector be 5, we need

$$\begin{aligned} \sqrt{3^2 + b^2} &= 5 \\ b^2 &= 25 - 9 = 16 \\ b &= \pm 4 \end{aligned}$$

Both $\vec{v}_1 = 3\vec{i} + 4\vec{j}$ and $\vec{v}_2 = 3\vec{i} - 4\vec{j}$ both have length 5, and the x component 3.

Problem code: YNSJK

55. The diagram below shows a rectangular box containing several vectors. Are the following statements true or false? Explain.

(a) $\vec{c} = \vec{f}$

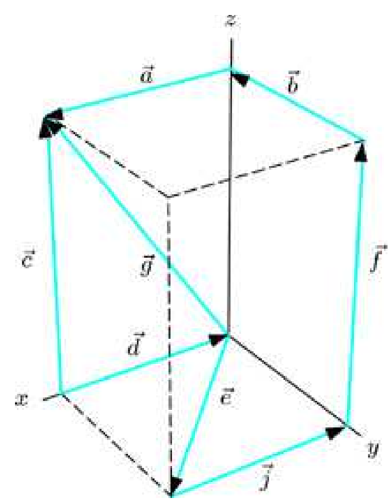
(b) $\vec{a} = \vec{d}$

(c) $\vec{a} = -\vec{b}$

(d) $\vec{g} = \vec{f} + \vec{a}$

(e) $\vec{e} = \vec{a} - \vec{b}$

(f) $\vec{d} = \vec{g} - \vec{c}$



(a) True, $\vec{c} = \vec{f}$. Sides of the rectangular box are parallel and the same length.

(b) False, since $\vec{a}$ and $\vec{d}$ point in opposite directions. Instead, we could say that $\vec{a} = -\vec{d}$.

(c) False. $-\vec{b}$ points in the direction opposite $\vec{b}$, but that will be perpendicular to $\vec{a}$.

(d) True. If we placed $\vec{f}$ on the z-axis (dark vertical line), then added $\vec{a}$, we would get $\vec{g}$.

(e) True. We move in the positive x-direction following vector $\vec{a}$ and then in the positive y-direction following vector $\vec{b}$. The resulting sum is the vector $\vec{e}$.

(f) False, vector $\vec{d}$ is the negative of the vector $\vec{g} - \vec{c}$. What would be true in this case is $\vec{d} = \vec{c} - \vec{g}$.

Problem code: JALHA (Video Solution by K. A.)

The Dot Product, Perpendicular and Parallel

For Problems 56-63, perform the following operations on the given 3-dimensional vectors.

$$\vec{a} = \langle 0,2,1\rangle\,,\quad \vec{b} = \langle -3,5,4\rangle\,,\quad \vec{c} = \langle 1,6,0\rangle$$
$$\vec{y} = \langle 4,-7,0\rangle\,,\quad \vec{z} = \langle 1,-3,-1\rangle$$

56. $\vec{a}\cdot\vec{y}$

$\vec{a}\cdot\vec{y} = (0)(4) + (2)(-7) + (1)(0) = -14$

Problem code: NBXQG

57. $\vec{c}\cdot\vec{y}$

$\vec{c}\cdot\vec{y} = (1)(4) + (6)(-7) + (0)(0) = -38$

Problem code: KAPSF

58. $\vec{a}\cdot\vec{b}$

$\vec{a}\cdot\vec{b} = (0)(-3) + (2)(5) + (1)(4) = 14$

Problem code: JWSNB

59. $\vec{a}\cdot\vec{z}$

$\vec{a}\cdot\vec{z} = (0)(1) + (2)(-3) + (1)(-1) = -7$

Problem code: UPSYK

60. $\vec{a}\cdot(\vec{c}+\vec{y})$

$\vec{a}\cdot(\vec{c}+\vec{y}) = \langle 0,2,1\rangle\cdot(\langle 1,6,0\rangle + \langle 4,-7,0\rangle) = \langle 0,2,1\rangle\cdot\langle 5,-1,0\rangle = (0)(5) + (2)(-1) + (1)(0) = -2$

Problem code: NQCAB

61. $\vec{c}\cdot\vec{a} + \vec{a}\cdot\vec{y}$

$\vec{c}\cdot\vec{a} + \vec{a}\cdot\vec{y} = \langle 1,6,0\rangle\cdot\langle 0,2,1\rangle + \langle 0,2,1\rangle\cdot\langle 4,-7,0\rangle = (0+12+0) + (0-14+0) = -2$

Problem code: DGWSM

62. $(\vec{a}\cdot\vec{b})\vec{a}$

$(\vec{a}\cdot\vec{b})\vec{a} = \underbrace{(\langle 0,2,1\rangle\cdot\langle -3,5,4\rangle)}_{\text{scalar}}\vec{a} = 14\vec{a} = 14\langle 0,2,1\rangle = \langle 0,28,14\rangle$

Problem code: ZFTUZ

63. $(\vec{a}\cdot\vec{y})(\vec{c}\cdot\vec{z})$

$(\vec{a}\cdot\vec{y})(\vec{c}\cdot\vec{z}) = (\langle 0,2,1\rangle\cdot\langle 4,-7,0\rangle)(\langle 1,6,0\rangle\cdot\langle 1,-3,-1\rangle) = (-14)(-17) = 238$

Problem code: JULYD

64. Let $\vec{v} = \langle 2,3\rangle$. Using only two-dimensional vectors, find a unit vector in the same direction as $\vec{v}$, and then find another vector perpendicular to $\vec{v}$.

To find the unit vector in the direction of $\vec{v} =< 2,3 >$, we find the length of $\vec{v}$, or $||\vec{v}|| = \sqrt{2^2+3^2} = \sqrt{13}$.

A unit vector in the same direction would be $\vec{u} = \frac{1}{\sqrt{13}} < 2,3 >$ or the equivalent $\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle$

A unit vector in the xy-plane perpendicular to v can be found by exchanging the components of $\vec{u}$ and making one negative:

$$\vec{n} = \left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right\rangle$$
$$\text{or}\quad \vec{n} = \left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right\rangle$$

The dot product of $\vec{n}$ with $\vec{v}$ would then be zero, indicating they are perpendicular.

Problem code: ECKAX

65. Which pairs (if any) of vectors from the following list

(a) Are perpendicular?

(b) Are parallel?

(c) Have an angle less than $\pi/2$ between them?

(d) Have an angle of more than $\pi/2$ between them?

$$\vec{a} = \langle 1,-3,-1\rangle\,,\vec{b} = \langle 1,1,2\rangle\,,$$
$$\vec{c} = \langle -2,-1,1\rangle\,,\vec{d} = \langle -1,-1,1\rangle$$

The angle between the vectors can be related to the dot product formula $||\vec{a}||\ ||\vec{b}||\cos(\theta)$.

- If the dot product is negative, then $\cos(\theta)$ must be negative, and so $\theta > \frac{\pi}{2}$;
- if the dot product is positive, $\theta < \frac{\pi}{2}$;
- if the dot product equals zero, it means the vectors are perpendicular (since none of the vectors are the zero vector).

$\vec{a} \cdot \vec{b} = -4$ angle greater than $\pi/2$
$\vec{a} \cdot \vec{c} = 0$ vectors are perpendicular
$\vec{a} \cdot \vec{d} = 1$ angle less than $\pi/2$
$\vec{b} \cdot \vec{c} = -1$ angle greater than $\pi/2$
$\vec{b} \cdot \vec{d} = 0$ vectors are perpendicular
$\vec{c} \cdot \vec{d} = 4$ angle less than $\pi/2$

Therefore, $\vec{a}, \vec{d}$ and $\vec{c}, \vec{d}$ have positive dot products, and therefore angles smaller than $\pi/2$ between them.

(d) Vectors with an angle of more than $\pi/2$ between them have a negative dot product, so pairs are $\vec{a}, \vec{b}$ and $\vec{b}, \vec{c}$. None of the vectors are multiples of each other, so none are parallel to each other.

Problem code: BVPKF

66. Which pairs of the vectors $\sqrt{3}\vec{i} + \vec{j}$, $3\vec{i} + \sqrt{3}\vec{j}$, $\vec{i} - \sqrt{3}\vec{j}$ are parallel and which are perpendicular?

This is easier to see in component format: $\langle \sqrt{3}, 1 \rangle$, $\langle 3, \sqrt{3} \rangle$, $\langle 1, -\sqrt{3} \rangle$.

The second is a multiple of the first: $\langle 3, \sqrt{3} \rangle = \sqrt{3} \langle \sqrt{3}, 1 \rangle$, so they are parallel.

The first and third are perpendicular, because their dot product is zero. Since the first and second are parallel, this means that the second and third are also perpendicular (which we can also verify by computing the dot product, which again equals zero).

Problem code: WSZUR

67. Compute the angle between the vectors $\vec{i}+\vec{j}+\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$.

Let $\vec{a}$ and $\vec{b}$ be the names of the vectors. Since $\vec{a} \cdot \vec{b} = \cos(\theta)||\vec{a}||\ \ ||\vec{b}||$,

$$\begin{aligned}\cos(\theta) &= \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||\ \ ||\vec{b}||} \\ &= \frac{1-1-1}{\sqrt{3}\sqrt{3}} \\ &= \frac{-1}{3}\end{aligned}$$

So $\theta = \arccos\left(\frac{-1}{3}\right) \approx 1.91$ radians $\approx 109.5^o$ degrees

Problem code: FPAPZ

68. (a) Give a 2-dimensional vector that is parallel to, but not equal to, $\vec{v} = \langle 4, 3 \rangle$.
(b) Give a vector that is perpendicular to $\vec{v}$.

(a) Any multiple of $\vec{v}$ would do: e.g. $\vec{u} = \langle 12, 9 \rangle = 3\vec{v}$

(b) We need the dot-product to be zero. The easiest example is to swap the components of $\vec{v}$ and make one negative: $\vec{u} = \langle -3, 4 \rangle$.

Check: $\vec{u} \cdot \vec{v} = \langle -3, 4 \rangle \cdot \langle 4, 3 \rangle = -12 + 12 = 0$. Since their dot product is zero, the two vectors must be perpendicular.

Problem code: RZEEC

69. For what values of t are $\vec{u} = \langle t, -1, 1 \rangle$ and $\vec{v} = \langle t, t, -2 \rangle$ perpendicular? Are there values of t for which $\vec{u}$ and $\vec{v}$ are parallel?

For $\vec{u}$ and $\vec{v}$ to be perpendicular, their dot product must be zero:

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle t, -1, 1 \rangle \cdot \langle t, t, -2 \rangle \\ &= t^2 - t - 2\end{aligned}$$

Setting the dot product equal to zero,

$$0 = t^2 - t - 2 = (t-2)(t+1)$$
$t = 2, -1$ will make $\vec{u}$ and $\vec{v}$ perpendicular

For the two to be parallel, they must be multiples of one another. I.e. there must be a multiplier λ such that $\vec{u} = \lambda\vec{v}$. We would need $\lambda = 1$ for the $\vec{i}$ components to be equal ($t = \lambda t$), and we would need $\lambda = \frac{1}{-2}$ for the $\vec{j}$ components to be equal ($1 = \lambda(-2)$). As a result, regardless of the value of t, the two vectors will never be parallel.

Problem code: LWMSH (Video Solution by N.P.)